Upper Bound

Elliot Krop* Irina Krop†

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We show the improved upper bound for the bipartite problem, when the two parts of G are of equal size.

Theorem 0.1

$$f(K_{n,n}, G_4, 3) \leq n, \textit{ for all } n \geq 3$$

0.1 The Coloring

We will explore the matrix

$$G = \begin{pmatrix} 1 & 2 & 3 & . & r & . & c+1 & . & n \\ 3 & 1 & 2 & . & r-1 & . & c & . & n-1 \\ v_3 & n-1 & 1 & . & r-2 & . & c-1 & . & n-2 \\ . & . & . & . & . & . & . & . & . \\ v_{n+1-r} & r+1 & r+2 & . & 1 & . & r+c & . & r \\ . & . & . & . & . & . & . & . & . \\ v_{n-1} & 3 & 4 & . & r+1 & . & c+2 & . & 2 \\ n-2 & u_2 & u_3 & . & u_r & . & u_{c+1} & . & 1 \end{pmatrix}$$

The values of v_i and u_i will be defined shortly.

Let permutation σ be the $n-1$ cycle $(1\ 2\ \dots\ n-1)$. That is, σ sends i to $i+1 \pmod{n-1}$. For a natural number m we shall write $m \pmod{n-1}$ for its representative in $\{1, 2, \dots, n-1\}$. For each r we define $\sigma^{(r)}$ by the rule $\sigma^{(r)}(c) \equiv r+c \pmod{n-1}$. Let us start with the matrix

$$C = \begin{pmatrix} 2 & 3 & . & c+1 & . & n \\ \sigma^0(1) & \sigma^0(2) & . & \sigma^0(c) & . & \sigma^0(n-1) \\ \sigma^{n-2}(1) & \sigma^{n-2}(2) & . & \sigma^{n-2}(c) & . & \sigma^{n-2}(n-1) \\ . & . & . & . & . & . \\ \sigma^r(1) & \sigma^r(2) & . & \sigma^r(c) & . & \sigma^r(n-1) \\ . & . & . & . & . & . \\ \sigma^2(1) & \sigma^2(2) & . & \sigma^2(c) & . & \sigma^2(n-1) \end{pmatrix}$$

We define matrix G by adding the first column $V = \{v_1, \dots, v_{n-1}, vu\}$ and the last row $U = \{vu, u_2, \dots, u_n\}$ to the matrix C.

*Department of Mathematics, Clayton State University, (ElliotKrop@clayton.edu)
†Depaul University, (irina.krop@gmail.com)

$$G=\begin{pmatrix} v_1 & 2 & 3 & . & c+1 & . & n \\ v_2 & \sigma^0(1) & \sigma^0(2) & . & \sigma^0(c) & . & \sigma^0(n-1) \\ v_3 & \sigma^{n-2}(1) & \sigma^{n-2}(2) & . & \sigma^{n-2}(c) & . & \sigma^{n-2}(n-1) \\ . & . & . & . & . & . & . \\ v_{n+1-r} & \sigma^r(1) & \sigma^r(2) & . & \sigma^r(c) & . & \sigma^r(n-1) \\ . & . & . & . & . & . & . \\ v_{n-1} & \sigma^2(1) & \sigma^2(2) & . & \sigma^2(c) & . & \sigma^2(n-1) \\ vu & u_2 & u_3 & . & u_{c+1} & . & u_n \end{pmatrix}$$

The entries of G will be defined as follows: for every 4-tuple $(i,j;l,m)$ with $1\le i<j\le n$ and $1\le l<m\le n$ the (2×2) matrix

$$G(i,j;l,m)=\begin{pmatrix} a_{il} & a_{im} \\ a_{jl} & a_{jm} \end{pmatrix}$$

We consider the colorings for the edges V and U in three types of even $n \pmod 6$.

Type 1: *Matrix $G_1=G$ for $n\equiv 2 \pmod 6$;* $[\,n=2+6k,\ k\ge 1\,]$

$$a_{i,1}=\begin{cases} 1, & i=1 \\ 3, & i=2 \\ n, & 3\le i\le \frac{n}{2}+1 \\ 2(i-1)-n, & \frac{n}{2}+2\le i\le n-1 \\ n-2, & i=n \end{cases}$$

$$a_{n,l}=\begin{cases} n-2l, & 1\le l\le \frac{n}{2}-1 \\ n, & \frac{n}{2}\le l\le n-2 \\ n-1, & l=n-1 \\ 1, & l=n \end{cases}$$

Type 2: *Matrix $G_2=G$ for $n\equiv 6 \pmod 6$;* $[\,n=6+6k,\ k\ge 1\,]$
We define Y as $\frac{n}{2}-2$ for even k, and as $\frac{n}{2}+1$ for odd k.

$$a_{i,1}=\begin{cases} 1, & i=1 \\ 3, & i=2 \\ n, & 3\le i\le \frac{n}{2}+1 \\ Y, & i=\frac{n}{2}+2 \\ 2(i-2)-n, & \frac{n}{2}+3\le i\le n-1 \\ n-2, & i=n \end{cases}$$

$$a_{n,l}=\begin{cases} n-2, & l=1 \\ n-2(l+1), & 2\le l\le \frac{n}{2}-2 \\ Y, & l=\frac{n}{2}-1 \\ n, & \frac{n}{2}\le l\le n-2 \\ n-1, & l=n-1 \\ 1, & l=n \end{cases}$$

Exception for $n=6$; $[k=0]$ the first row $V=\{1,5,6,6,4,\}$, the last column $U=\{3,6,6,6,5,1\}$.

Type 3: *Matrix* $G_3 = G$ *for* $n \equiv 4 \pmod 6$; $\quad [\, n = 4 + 6k,\ k \geq 4\,]$
The regularity starts with $n > 22$.

$$a_{i,1} = \begin{cases} 1, & i = 1 \\ 3, & i = 2 \\ n, & 3 \leq i \leq \frac{n}{2} + 1 \\ n - 9, & i = \frac{n}{2} + 2 \\ 2(i-2) - n, & \frac{n}{2} + 3 \leq i \leq \frac{5n+4}{6} \\ 2(i-1) - n, & \frac{5n+10}{6} \leq i \leq n - 1 \\ n - 2, & i = n \end{cases}$$

$$a_{n,l} = \begin{cases} n - 2l, & 1 \leq l \leq \frac{n-4}{6} \\ n - 2(l+1), & \frac{n+2}{6} \leq l \leq \frac{n}{2} - 2 \\ n - 9, & l = \frac{n}{2} - 1 \\ n, & \frac{n}{2} \leq l \leq n - 2 \\ n - 1, & l = n - 1 \\ 1, & l = n \end{cases}$$

Exceptions:
For $n = 10$ we replace $(n - 9)$ with $(n - 8)$.
For $n = 16$ we replace $(n - 9)$ with $(n - 11)$.
For $n = 22$ we replace $(n - 9)$ with $(n - 5)$ and the definitions:

$$a_{i,1} = \begin{cases} 2(i-2) - n, & \frac{n}{2} + 3 \leq i \leq \frac{5n-2}{6} \\ 2(i-1) - n, & \frac{5n+4}{6} \leq i \leq n - 1 \end{cases}$$

$$a_{n,l} = \begin{cases} n - 2l, & 1 \leq l \leq \frac{n-10}{6} \\ n - 2(l+1), & \frac{n-4}{6} \leq l \leq \frac{n}{2} - 2 \end{cases}$$

0.2 Proof in Steps

Proof.

First, we show that every $4 - \textit{cycle}$ defined in the *basic coloring* is *almost rainbow*. That is, given $i < j$ and $l < m$ we show that $a_{i,l}, a_{j,l}, a_{i,m}, a_{j,m}$ contains at least three distinct elements in the *basic coloring*.

Step 1:

We start with the matrix C and look at two occurrences, which are identical for each of the types of even $n \pmod 6$ specified above.

Case 1: We take the submatrix of $G(i, j; l, m)$ with $2 \leq l < m \leq n$, $2 \leq i < j < n$, and let $s = (n+1) - i$, $t = (n+1) - j$. A typical (2×2) submatrix in this case has the form:

$$\begin{pmatrix} \sigma^s(l-1) & \sigma^s(m-1) \\ \sigma^t(l-1) & \sigma^t(m-1) \end{pmatrix}$$

We wish to show there are three distinct elements: $\sigma^s(l-1) \neq \sigma^t(l-1), \sigma^t(l-1) \neq \sigma^t(m-1)$, $\sigma^s(l-1) \neq \sigma^t(m-1)$. Suppose $\sigma^{(s)}(l-1) \equiv \sigma^{(t)}(l-1) \Rightarrow s \equiv t$

(mod $n-1$), which is a contradiction. Suppose $\sigma^{(t)}(l-1) \equiv \sigma^{(t)}(m-1) \Rightarrow l \equiv m$ (mod $n-1$), which is a contradiction. Suppose $\sigma^{(s)}(l-1) \equiv \sigma^{(t)}(m-1) \Rightarrow s+l \equiv t+m \pmod{n-1}$, and assume there are three distinct elements: $\sigma^t(l-1) \neq \sigma^t(m-1)$, $\sigma^s(m-1) \neq \sigma^t(m-1)$, $\sigma^s(m-1) \neq \sigma^t(l-1)$. Follow the argument above the first two inequalities are correct. Suppose $\sigma^s(m-1) \equiv \sigma^t(l-1) \Rightarrow s+m \equiv t+l \pmod{n-1}$. Subtracting equations $s+l \equiv t+m$ and $s+m \equiv t+l \Rightarrow l \equiv m$ (mod $n-1$), which is a contradiction. One of the following two sets has three distinct elements: $\{\sigma^s(l-1),\ \sigma^t(l-1),\ \sigma^t(m-1)\}$ or $\{\sigma^t(l-1),\ \sigma^t(m-1),\ \sigma^s(m-1)\}$.

Case 2: We take the submatrix of $G(i,j;l,m)$ with $2 \leq l < m \leq n$, $i=1$, $1 < j < n$, and let $r = (n+1)-j$. A typical (2×2) submatrix has the form:

$$\begin{pmatrix} l & m \\ \sigma^r(l-1) & \sigma^r(m-1) \end{pmatrix}$$

We wish to show there are three distinct elements: $l \neq m$, $m \neq \sigma^r(m-1)$, $l \neq \sigma^r(m-1)$. Suppose $m \equiv \sigma^{(r)}(m-1) \Rightarrow r \equiv 1 \pmod{n-1}$, which is a contradiction. Suppose $l \equiv \sigma^{(r)}(m-1) \Rightarrow l \equiv r+m-1 \pmod{n-1}$, and assume there are three distinct elements: $\sigma^r(l-1) \neq \sigma^r(m-1)$, $m \neq \sigma^r(m-1)$, $m \neq \sigma^r(l-1)$. The first two inequalities are correct. Suppose $m \equiv \sigma^r(l-1) \Rightarrow m \equiv r+l-1 \pmod{n-1}$. Subtracting equations $l \equiv r+m-1$ and $m \equiv r+l-1 \Rightarrow r = 1$ (mod $n-1$), which is a contradiction. One of the following two sets has three distinct elements: $\{l,\ m,\ \sigma^r(m-1)\}$ or $\{\sigma^r(l-1),\ \sigma^r(m-1),\ m\}$.

Step 2:

For matrix $G(i,j;l,m)$ with $i=1$, $j=n$ and $2 \leq l < m \leq n$ we look at five cases and consider every matrix type defined above of even $n \pmod 6$.

Case 1: We take $G(i,j;l,m)$ with $2 \leq l < m \leq \frac{n}{2}-1$, $i=1$, $j=n$.

Consider G_1.

$$\begin{pmatrix} l & m \\ n-2l & n-2m \end{pmatrix}$$

We wish to show there are three distinct entries: $l \neq n-2l$, $n-2l \neq n-2m$, $l \neq n-2m$. Suppose $l = n-2l \Rightarrow 3l = n$ and since $n = 2+6k$ this is a contradiction. Suppose $n-2l = n-2m \Rightarrow l = m$, which is a contradiction. Suppose $l = n-2m$ and we wish to show there are three distinct elements: $l \neq m$, $l \neq n-2l$, $m \neq n-2l$. As shown above the first inequality is correct. Suppose $m = n-2l$ and since $l = n-2m \Rightarrow l = m$, which is a contradiction. One of the following two sets has three distinct elements: $\{l, n-2l, n-2m\}$ or $\{l, m, n-2l\}$.

Consider G_2

1. G_2 with $2 \leq l < m \leq \frac{n}{2}-2$, $i=1$, $j=n$.

$$\begin{pmatrix} l & m \\ n-2(l+1) & n-2(m+1) \end{pmatrix}$$

We wish to show there are three distinct entries: $l \neq n-2(m+1)$, $l \neq n-2(l+1)$, $n-2(l+1) \neq n-2(m+1)$. Suppose $l = n-2l-2 \Rightarrow 3l = n-2$ and since $n = 6+6k$ this is a contradiction. Suppose $n-2l = n-2m \Rightarrow l = m$, which is a contradiction.

Suppose $l = n - 2(m+1)$ and we wish to show there are three distinct elements: $l \neq m$, $l \neq n-2(l+1)$, $m \neq n-2(l+1)$. As shown above the first inequality is correct. Suppose $m = n-2(l+1)$ and since $l = n-2(m+1) \Rightarrow l = m$, which is a contradiction. One of the following two sets has three distinct elements: $\{l, n-2(m+1), n-2(l+1)\}$ or $\{l, m, n-2(l+1)\}$.

2. G_2 with $2 \leq l \leq \frac{n}{2} - 2$, $m = \frac{n}{2} - 1$, $i = 1$, $j = n$.

$$\begin{pmatrix} l & m \\ n-2(l+1) & Y \end{pmatrix}$$

If K *is Even* $\Rightarrow m = \frac{n}{2} - 1$, $Y = \frac{n}{2} - 2 \Rightarrow Y = m - 1$.

We wish to show there are three distinct entries: $l \neq m$, $m \neq m-1$, $l \neq m-1$. Assume $l = m-1$ and we wish to show there are three distinct entries: $m \neq m-1$, $m \neq n-2(l+1)$, $m-1 \neq n-2(l+1)$. Suppose $m = n-2(l+1)$ and since $l = m-1$ and $m = \frac{n}{2} - 1 \Rightarrow n = 6$, which is a contradiction. Suppose $m-1 = n-2(l+1)$ and since $m = \frac{n}{2} - 1$ and $l = m-1 \Rightarrow n = 8$, which is a contradiction. One of the following two sets has three distinct elements: $\{l, m, Y\}$ or $\{m, Y, n-2(l+1)\}$.

If K *is Odd* $\Rightarrow m = \frac{n}{2} - 1$, $Y = \frac{n}{2} + 1 \Rightarrow Y = m + 2$. Three distinct elements are $\{l, m, Y\}$.

Consider G_3

1. G_3 with $i = 1$, $j = n$ and $(2 \leq l < m \leq \frac{n-4}{6}$ or $\frac{n+2}{6} \leq l < m \leq \frac{n}{2} - 2)$. The argument is similar to above one with G_1. One of the following two sets has three distinct elements: $\{l, n-2l, n-2m\}$ or $\{l, m, n-2l\}$.

2. G_3 with $2 \leq l \leq \frac{n-4}{6}$, $\frac{n+2}{6} \leq m \leq \frac{n}{2} - 2$, $i = 1$, $j = n$.

$$\begin{pmatrix} l & m \\ n-2l & n-2(m+1) \end{pmatrix}$$

We wish to show there are three distinct entries: $l \neq n-2l$, $n-2l \neq n-2(m+1)$, $l \neq n-2(m+1)$. Suppose $l = n-2l \Rightarrow 3l = n$ and since $n = 4+6k$ this is a contradiction. Suppose $n-2l = n-2(m+1) \Rightarrow l = m+1$, which is a contradiction. Suppose $l = n-2(m+1)$ and we wish to show there are three distinct elements: $l \neq m$, $l \neq n-2l$, $m \neq n-2l$. The first two inequalities are correct. Suppose $m = n-2l$ and since $l = n-2(m+1) \Rightarrow m = l-2$, which is a contradiction. One of the following two sets has three distinct elements: $\{l, n-2l, n-2(m+1))\}$ or $\{l, m, n-2l\}$.

3. G_3 with $2 \leq l \leq \frac{n-4}{6}$, $m = \frac{n}{2} - 1$, $i = 1$, $j = n$.

$$\begin{pmatrix} l & m \\ n-2l & n-9 \end{pmatrix}$$

We wish to show there are three distinct entries: $l \neq m$, $m \neq n-9$, $l \neq n-9$. Suppose $l = n-9$ and since $l < \frac{n-4}{6} \Rightarrow n-9 < \frac{n-4}{6} \Rightarrow n < 10$, which is a contradiction. Suppose $m = n-9 \Rightarrow \frac{n}{2} - 1 = n-9 \Rightarrow n = 16$, which is a contradiction. There are three distinct elements $\{l, m, n-9\}$.

4. G_3 with $\frac{n+2}{6} \leq l \leq \frac{n}{2} - 2$, $m = \frac{n}{2} - 1$, $i = 1$, $j = n$.

$$\begin{pmatrix} l & m \\ n-2(l+1) & n-9 \end{pmatrix}$$

We wish to show there are three distinct entries: $l \neq m$, $m \neq n-9$, $l \neq n-9$. Suppose $l = n-9$ and since $l < \frac{n}{2} - 2 \Rightarrow n - 9 < \frac{n}{2} - 2 \Rightarrow n < 14$, which is a contradiction. Suppose $m = n - 9 \Rightarrow \frac{n}{2} - 1 = n - 9 \Rightarrow n = 16$, which is a contradiction. There are three distinct elements $\{l, m, n-9\}$.

Exceptions: For $n = 16$ three distinct elements are $\{14 - 2l, 5, 7\}$. For $n = 22$ three distinct elements are $\{l, m, 17\}$

Case 2: For matrix $G(i,j;l,m)$ with $i = 1$, $j = n$ and ($\frac{n}{2} \leq l < m \leq n-2$ or $2 \leq l \leq \frac{n}{2} - 1$, $\frac{n}{2} \leq m \leq n-2$) three distinct elements are $\{l, m, n\}$.

Case 3: We take the submatrix $G(i,j;l,m)$ with $2 \leq l \leq \frac{n}{2} - 1$, $m = n-1$, $i = 1$, $j = n$.

Consider G_1

$$\begin{pmatrix} l & m \\ n-2l & n-1 \end{pmatrix}$$

We wish to show there are three distinct entries: $l \neq n-1$, $n - 2l \neq n-1$, $l \neq n - 2l$. Suppose $n - 2l = n - 1 \Rightarrow l = \frac{1}{2}$, which is a contradiction. Suppose $l = n - 2l \Rightarrow 3l = n$ and since $n = 2 + 6k$ this is a contradiction. There are three distinct elements $\{l, n-1, n-2l\}$.

Consider G_2

1. G_2 with $2 \leq l \leq \frac{n}{2} - 2$, $m = n-1$, $i = 1$, $j = n$.

$$\begin{pmatrix} l & m \\ n-2(l+1) & n-1 \end{pmatrix}$$

We wish to show there are three distinct entries: $l \neq n-1$, $n-2(l+1) \neq n-1$, $l \neq n-2(l+1)$. Suppose $n - 2(l+1) = n-1 \Rightarrow l = -\frac{1}{2}$, which is a contradiction. Suppose $l = n-2(l+1) \Rightarrow 3l = n-2$ and since $n = 6+6k$ this is a contradiction. There are three distinct elements $\{l, n-1, n-2(l+1)\}$.

2. G_2 with $l = \frac{n}{2} - 1$, $m = n-1$, $i = 1$, $j = n$.

$$\begin{pmatrix} l & m \\ Y & n-1 \end{pmatrix}$$

If K *is Even* $\Rightarrow Y = \frac{n}{2} - 2$. If $n - 1 = Y \Rightarrow n - 1 = \frac{n}{2} - 2 \Rightarrow n = -2$, which is a contradiction. Three distinct entries are $\{l, Y, n-1\}$. If K *is Odd* $\Rightarrow Y = \frac{n}{2} + 1$, and three distinct entries are $\{l, Y, n-1\}$.

Consider G_3

1. For G_3 with $l \leq \frac{n-4}{6}$, $m = n-1$, $i = 1$, $j = n$ three distinct elements are $\{l, n-1, n-2l\}$ (similar to G_1.)

2. For G_3 with $\frac{n+2}{6} \leq l < n-1$, $m = n-1$, $i = 1$, $j = n$.

$$\begin{pmatrix} l & m \\ n-2(l+1) & n-1 \end{pmatrix}$$

We wish to show there are three distinct entries: $l \neq n-1$, $n-2(l+1) \neq n-1$, $l \neq n-2(l+1)$. Suppose $n-2(l+1) = n-1 \Rightarrow l = -\frac{1}{2}$, which is a contradiction.

Suppose $l = n - 2(l+1) \Rightarrow 3l = n - 2$ and since $n = 4 + 6k$ this is a contradiction. Three distinct elements are $\{l, n-1, n-2(l+1)\}$.

3. For G_3 with $l = \frac{n}{2} - 1$, $m = n - 1$, $i = 1$, $j = n$ three distinct entries are $\{l, n-9, n-1)\}$.

Exceptions: For $n = 16$ three distinct entries are $\{7, 5, 15\}$. For $n = 22$ three distinct entries are $\{10, 17, 21\}$.

Case 4: For the submatrix $G(i,j;l,m)$ with $\frac{n}{2} \leq l \leq n-2$, $m = n-1$, $i = 1$, $j = n$ three distinct elements are $\{l, n, n-1\}$.

Case 5: For the submatrix $G(i,j;l,m)$ with $l = n-1$, $m = n$, $i = 1$, $j = n$ three distinct elements are $\{n-1, n, 1\}$.

Step 3:

For matrix $G(i,j;l,m)$ with $2 \leq i \leq n-1$, $j = n$, $2 \leq l < m \leq n$ we look at five cases and consider every matrix type defined above of even $n \pmod 6$.

Case 1: We take $G(i,j;l,m)$.
Consider G_1 with $2 \leq l < m \leq \frac{n}{2} - 1$, $2 \leq i \leq n-1$, $j = n$.

$$\begin{pmatrix} \sigma^s(l-1) & \sigma^s(m-1) \\ n-2l & n-2m \end{pmatrix}$$

We wish to show there are three distinct entries: $\sigma^s(l-1) \neq n-2l$, $\sigma^s(l-1) \neq n-2m$, $n-2l \neq n-2m$. Suppose $n-2l = n-2m \Rightarrow m = l$, which is a contradiction. Suppose $\sigma^s(l-1) \equiv n-2l$ and we wish to show there are three distinct entries: $n-2m \neq n-2l$, $\sigma^s(m-1) \neq n-2l$, $\sigma^s(m-1) \neq n-2m$. The first statement is correct. Since $\sigma^s(l-1) \equiv n-2l$ and $\sigma^s(l-1) \neq \sigma^s(m-1) \Rightarrow \sigma^s(m-1) \neq n-2l$. Subtracting equations $\sigma^s(m-1) \equiv n-2m$ and $\sigma^s(l-1) \equiv n-2l \Rightarrow m \equiv l \pmod{n-1}$, which is a contradiction. Suppose $\sigma^s(l-1) \equiv n-2m$ and we wish to show there are three distinct entries: $n-2m \neq n-2l$, $\sigma^s(m-1) \neq n-2m$, $\sigma^s(m-1) \neq n-2l$. The statement $n-2m \neq n-2l$ is correct. Since $\sigma^s(l-1) \equiv n-2m$ and $\sigma^s(l-1) \neq \sigma^s(m-1) \Rightarrow \sigma^s(m-1) \neq n-2m$. Subtracting equations $\sigma^s(m-1) \equiv n-2l$ and $\sigma^s(l-1) \equiv n-2m \Rightarrow m \equiv l \pmod{n-1}$, which is a contradiction. One of the following two sets has three distinct elements: $\{\sigma^s(l-1), n-2l, n-2m\}$ or $\{\sigma^s(m-1), n-2l, n-2m\}$.

Consider G_2

1. G_2 with $2 \leq l < m \leq \frac{n}{2} - 2$, $2 \leq i \leq n-1$, $j = n$.

$$\begin{pmatrix} \sigma^s(l-1) & \sigma^s(m-1) \\ n-2(l+1) & n-2(m+1) \end{pmatrix}$$

We wish to show there are three distinct entries: $\sigma^s(l-1) \neq n-2(l+1)$, $\sigma^s(l-1) \neq n-2(m+1)$, $n-2(l+1) \neq n-2(m+1)$. Suppose $n-2(l+1) = n-2(m+1) \Rightarrow m = l$, which is a contradiction. Suppose $\sigma^s(l-1) \equiv n-2(l+1)$ and we wish to show there are three distinct entries: $n-2(m+1) \neq n-2(l+1)$, $\sigma^s(m-1) \neq n-2(l+1)$, $\sigma^s(m-1) \neq n-2(m+1)$. The first statement is correct. Since $\sigma^s(l-1) \equiv n-2(l+1)$ and $\sigma^s(l-1) \neq \sigma^s(m-1) \Rightarrow \sigma^s(m-1) \neq n-2(l+1)$. Subtracting equations $\sigma^s(m-1) \equiv n-2(m+1)$ and $\sigma^s(l-1) \equiv n-2(l+1) \Rightarrow m \equiv l \pmod{n-1}$, which is a contradiction.

Assume $\sigma^s(l-1) \equiv n-2(m+1)$ and we wish to show there are three distinct entries: $n-2(m+1) \neq n-2(l+1)$, $\sigma^s(m-1) \neq n-2(m+1)$, $\sigma^s(m-1) \neq n-2(l+1)$ The first statement is correct. Since $\sigma^s(l-1) \equiv n-2(m+1)$ and $\sigma^s(l-1) \neq \sigma^s(m-1) \Rightarrow \sigma^s(m-1) \neq n-2(m+1)$. Subtracting equations $\sigma^s(m-1) \equiv n-2(l+1)$ and $\sigma^s(l-1) \equiv n-2(m+1) \Rightarrow m \equiv l \pmod{n-1}$, which is a contradiction. One of the following two sets has three distinct elements: $\{\sigma^s(l-1), n-2(l+1), n-2(m+1)\}$ or $\{\sigma^s(m-1), n-2(l+1), n-2(m+1)\}$.

2. G_2 with $2 \leq l \leq \frac{n}{2}-2$, $m = \frac{n}{2}-1$, $2 \leq i \leq n-1$, $j = n$.

$$\begin{pmatrix} \sigma^s(l-1) & \sigma^s(m-1) \\ n-2(l+1) & Y \end{pmatrix}$$

If K *is Even* $\Rightarrow Y = \frac{n}{2}-2$. We wish to show there are three distinct entries: $\sigma^s(l-1) \neq \sigma^s(\frac{n}{2}-2)$, $\sigma^s(\frac{n}{2}-2) \neq \frac{n}{2}-2$, $\sigma^s(l-1) \neq \frac{n}{2}-2$. Two first inequalities are correct. Assume $\sigma^s(l-1) \equiv \frac{n}{2}-2$ and we wish to show there are three distinct entries: $\sigma^s(\frac{n}{2}-2) \neq (\frac{n}{2}-2)$, $n-2(l+1) \neq \frac{n}{2}-2$, $\sigma^s(\frac{n}{2}-2) \neq n-2(l+1)$. Suppose $n-2(l+1) = \frac{n}{2}-2 \Rightarrow n = 4l$. Hence $n = 6(k+1) \Rightarrow 4l = 6(k+1) \Rightarrow l = 1.5(k+1)$, and since k is even $\Rightarrow l$ is not integer, which is a contradiction. Subtracting equations $\sigma^s(\frac{n}{2}-2) \equiv n-2(l+1)$ and $\sigma^s(l-1) \equiv \frac{n}{2}-2 \Rightarrow l \equiv 1$, which is a contradiction. One of the following two sets has three distinct elements: $\{\sigma^s(l-1), \sigma^s(m-1), \frac{n}{2}-2\}$ or $\{\sigma^s(m-1), n-2(l+1), \frac{n}{2}-2\}$.

If K *is Odd* $\Rightarrow Y = \frac{n}{2}+1$. We wish to show there are three distinct entries: $\sigma^s(m-1) \neq Y$, $\sigma^s(m-1) \neq n-2(l+1)$, $n-2(l+1) \neq Y$. Suppose $n-2(l+1) = Y \Rightarrow 4l = n-6$. Since $n = 6k+6 \Rightarrow l = 1.5k$ and since k is odd l is not integer, this is a contradiction. Suppose $\sigma^s(m-1) \equiv Y \Rightarrow s + \frac{n}{2} - 2 \equiv \frac{n}{2}+1 \Rightarrow s \equiv 3 \pmod{n-1}$. Suppose $\sigma^s(m-1) \equiv n-2(l+1) \Rightarrow s \equiv \frac{n}{2}-2l \pmod{n-1}$. We wish to show there are three distinct entries: $\sigma^s(l-1) \neq n-2(l+1)$, $n-2(l+1) \neq \frac{n}{2}+1$, $\sigma^s(l-1) \neq \frac{n}{2}+1$. We already know that $n-2(l+1) \neq \frac{n}{2}+1$. Suppose $\sigma^s(l-1) \equiv \frac{n}{2}+1 \Rightarrow 2s \equiv n+4-2l \pmod{n-1}$. Suppose $\sigma^s(l-1) \equiv n-2(l+1) \Rightarrow 3l+s \equiv n-1 \pmod{n-1}$. Since $s \equiv 3$ from the equation $2s \equiv n+4-2l \Rightarrow l \equiv \frac{n}{2}-1$, which is a contradiction. And from the equation $s+3l \equiv n-1$ since $n = 6k+6 \Rightarrow l \equiv 2k+\frac{2}{3}$, which is a contradiction. Going back to the assumption $2s \equiv n-4l$ and subtracting it from $2s \equiv n+4-2l \Rightarrow l \equiv -2$, which is a contradiction. Subtracting $2s \equiv n-4l$ from $s+3l \equiv n-1 \Rightarrow l \equiv \frac{n}{2}-1$, which is a contradiction. One of the following two sets has three distinct elements: $\{Y, \sigma^s(l-1), n-2(l+1)\}$ or $\{\sigma^s(m-1), n-2(l+1), Y\}$.

Consider G_3.

1. G_3 with ($2 \leq l < m \leq \frac{n-4}{6}$ or $\frac{n+2}{6} \leq l < m \leq \frac{n}{2}-2$), $2 \leq i \leq n-1$, $j = n$. This case is similar to the case for the matrix G_1.

2. G_3 with $2 \leq l \leq \frac{n-4}{6}$, $\frac{n+2}{6} \leq m \leq \frac{n}{2}-2$, $2 \leq i \leq n-1$, $j = n$.

$$\begin{pmatrix} \sigma^s(l-1) & \sigma^s(m-1) \\ n-2l & n-2(m+1) \end{pmatrix}$$

We wish to show there are three distinct entries: $\sigma^s(l-1) \neq n-2l$, $\sigma^s(l-1) \neq n-2(m+1)$, $n-2l \neq n-2(m+1)$. Suppose $n-2l = n-2(m+1) \Rightarrow l = m+1$, which is a contradiction. Suppose $\sigma^s(l-1) \equiv n-2l$ and we wish to show there are three distinct entries: $n-2(m+1) \neq n-2l$, $\sigma^s(m-1) \neq n-2l$, $\sigma^s(m-1) \neq n-2(m+1)$. Since $\sigma^s(l-1) \equiv n-2l$ and $\sigma^s(l-1) \neq \sigma^s(m-1) \Rightarrow \sigma^s(m-1) \neq n-2l$.

Subtracting two equations $\sigma^s(m-1) \equiv n-2(m+1)$ and $\sigma^s(l-1) \equiv n-2l \Rightarrow m \equiv l-\frac{2}{3} \pmod{n-1}$, which is a contradiction. Going back to the assumption $\sigma^s(l-1) \equiv n-2m$ we wish to show there are three distinct entries: $n-2(m+1) \neq n-2l$, $\sigma^s(m-1) \neq n-2(m+1)$, $\sigma^s(m-1) \neq n-2l$. Since $\sigma^s(l-1) \equiv n-2(m+1)$ and $\sigma^s(l-1) \neq \sigma^s(m-1) \Rightarrow \sigma^s(m-1) \neq n-2(m+1)$. Subtracting two equations $\sigma^s(m-1) \equiv n-2l$ and $\sigma^s(l-1) \equiv n-2(m+1) \Rightarrow m \equiv l-2 \pmod{n-1}$, which is a contradiction. One of the following two sets has three distinct elements: $\{\sigma^s(l-1), n-2l, n-2(m+1)\}$ or $\{\sigma^s(m-1), n-2l, n-2(m+1)\}$.

3. G_3 with $2 \leq l \leq \frac{n-4}{6}$, $m = \frac{n}{2}-1$, $2 \leq i \leq n-1$, $j = n$

$$\begin{pmatrix} \sigma^s(l-1) & \sigma^s(m-1) \\ n-2l & n-9 \end{pmatrix}$$

We wish to show there are three distinct entries: $n-2l \neq n-9$, $n-2l \neq \sigma^s(m-1)$, $\sigma^s(m-1) \neq n-9$. Suppose $n-2l = n-9 \Rightarrow 2l = 9$, which is a contradiction. Suppose $\sigma^s(m-1) \equiv n-9 \Rightarrow s \equiv \frac{n}{2}-7 \pmod{n-1}$ or $s = \frac{n}{2}-7$. Suppose $\sigma^s(m-1) \equiv n-2l \Rightarrow s-2 \equiv \frac{n}{2}-2l \pmod{n-1}$ and since $l \leq \frac{n-4}{6} \Rightarrow \frac{n}{2}-2l > 0 \Rightarrow s-2 = \frac{n}{2}-2l$. We wish to show there are three distinct entries: $\sigma^s(l-1) \neq n-2l$, $\sigma^s(l-1) \neq n-9$, $n-2l \neq n-9$. Since $\sigma^s(m-1) \equiv n-2l$ and $\sigma^s(l-1) \neq \sigma^s(m-1) \Rightarrow \sigma^s(l-1) \neq n-2l$. Suppose $\sigma^s(l-1) \equiv n-9 \Rightarrow s+l \equiv n-8$ and we assumed $s-2 \equiv \frac{n}{2}-2l$. Subtracting them $\Rightarrow \frac{n}{2}+l-10 \equiv 0$. For type 3 matrix we have $n = 4+6k \Rightarrow l \equiv 8-3k$ and since $k \geq 4 \Rightarrow l < 0$, which is a contradiction. Assume that $s = n/2-7$ and we wish to show there are three distinct entries: $\sigma^s(l-1) \neq n-2l$, $\sigma^s(l-1) \neq n-9$, $n-2l \neq n-9$, Suppose $\sigma^s(l-1) \equiv n-9$ and since $s = n/2-7 \Rightarrow l = n/2-1$, this is a contradiction since $l \leq (n-4)/6$. Suppose $\sigma^s(l-1) \equiv n-2l$ and $s = n/2-7 \Rightarrow 3l \equiv n/2+8$. Since $n/2 = 2+3k \Rightarrow 3l \equiv 10+3k$, right part is not divisible by 3 this is a contradiction. One of the following two sets has three distinct elements: $\{\sigma^s(m-1), n-2l, n-9\}$ or $\{\sigma^s(l-1), n-2l, n-9\}$.

Exceptions: For $n = 16$ replacing $n-9$ with $n-11$ and following the argument above we get that one of the following two sets has three distinct elements: $\{\sigma^s(m-1), n-2l, n-11\}$ or $\{\sigma^s(l-1), n-2l, n-11\}$. For $n = 22$ replacing $n-9$ with $n-5$ and following the same argument we get that one of the following two sets has three distinct elements: $\{\sigma^s(m-1), n-2l, n-5\}$ or $\{\sigma^s(l-1), n-2l, n-5\}$.

4. G_3 with $\frac{n+2}{6} \leq l \leq \frac{n}{2}-2$, $m = \frac{n}{2}-1$, $2 \leq i \leq n-1$, $j = n$, and let $s = (n+1)-i$.

$$\begin{pmatrix} \sigma^s(l-1) & \sigma^s(m-1) \\ n-2(l+1) & n-9 \end{pmatrix}$$

We wish to show there are three distinct entries: $n-2l-2 \neq n-9$, $n-2l-2 \neq \sigma^s(m-1)$, $\sigma^s(m-1) \neq n-9$. Suppose $n-2l-2 = n-9 \Rightarrow 2l = 7$, which is a contradiction. Suppose $\sigma^s(m-1) \equiv n-9 \Rightarrow s \equiv \frac{n}{2}-7 \pmod{n-1}$ or $s = \frac{n}{2}-7$. Suppose $\sigma^s(m-1) \equiv n-2l-2 \Rightarrow s \equiv \frac{n}{2}-2l \pmod{n-1}$ and since $l \leq \frac{n-4}{6} \Rightarrow \frac{n}{2}-2l > 0 \Rightarrow s = \frac{n}{2}-2l$. We wish to show there are three distinct entries: $\sigma^s(l-1) \neq n-2l-2$, $\sigma^s(l-1) \neq n-9$, $n-2l-2 \neq n-9$. Since $\sigma^s(m-1) \equiv n-2l-2$ and $\sigma^s(l-1) \neq \sigma^s(m-1) \Rightarrow \sigma^s(l-1) \neq n-2l-2$. Suppose $\sigma^s(l-1) \equiv n-9 \Rightarrow s+l \equiv n-8$ and we assumed $s \equiv \frac{n}{2}-2l$. Subtracting them $\Rightarrow \frac{n}{2}+l-8 \equiv 0$. For type 3 matrix we have $n = 4+6k \Rightarrow l \equiv 6-3k$ and since

$k \geq 4 \Rightarrow l < 0$, which is a contradiction. Assume that $s = n/2 - 7$ and we wish to show there are three distinct entries: $\sigma^s(l-1) \neq n-2l-2$, $\sigma^s(l-1) \neq n-9$, $n-2l-2 \neq n-9$, Suppose $\sigma^s(l-1) \equiv n-9$ and since $s = n/2-7 \Rightarrow l = n/2-1$, this is a contradiction since $l \leq (n-4)/6$. Suppose $\sigma^s(l-1) \equiv n-2l-2$ and $s = n/2-7 \Rightarrow 3l \equiv n/2+6$. Since $n/2 = 2+3k \Rightarrow 3l \equiv 8+3k$, right part is not divisible by 3 this is a contradiction. One of the following two sets has three distinct elements: $\{\sigma^s(m-1), n-2l-2, n-9\}$ or $\{\sigma^s(l-1), n-2l-2, n-9\}$.

Exceptions: For $n = 16$ replacing $n-9$ with $n-11$ and following the argument above we get that one of the following two sets has three distinct elements: $\{\sigma^s(m-1), n-2l-2, n-11\}$ or $\{\sigma^s(l-1), n-2l-2, n-11\}$. For $n = 22$ replacing $n-9$ with $n-5$ and following the same argument we get that one of the following two sets has three distinct elements: $\{\sigma^s(m-1), n-2l-2, n-5\}$ or $\{\sigma^s(l-1), n-2l-2, n-5\}$.

Case 2: We take matrix $G(i,j;l,m)$ with $(2 \leq l \leq \frac{n}{2}-1,\ \frac{n}{2} \leq m \leq n-2)$, or $(\frac{n}{2} \leq l < m \leq n-2)$ and $2 \leq i \leq n-1$, $j = n$. Three distinct entries are $\{\sigma^s(l-1), \sigma^s(m-1), n\}$

Case 3: We take matrix $G(i,j;l,m)$ with $2 \leq l \leq \frac{n}{2}-1$, $m = n-1$, $2 \leq i \leq n-1$, $j = n$.
Consider G_1.

$$\begin{pmatrix} \sigma^s(l-1) & \sigma^s(m-1) \\ n-2l & n-1 \end{pmatrix}$$

We wish to show there are three distinct entries: $n-2l \neq n-1$, $n-2l \neq \sigma^s(n-2)$, $\sigma^s(n-2) \neq n-1$. Suppose $n-2l = n-1 \Rightarrow l = \frac{1}{2}$, which is a contradiction. Suppose $n-2l \equiv \sigma^s(n-2) \Rightarrow n-2l \equiv s+n-2 \pmod{n-1} \Rightarrow s+2l \equiv 2$. Suppose $n-1 \equiv \sigma^s(n-2) \Rightarrow s \equiv 1 \Rightarrow i \equiv n$, which is a contradiction. Assume $s+2l \equiv 2$ and we wish to show there are three distinct entries: $\sigma^s(l-1) \neq \sigma^s(n-2)$, $\sigma^s(l-1) \neq n-1$, $\sigma^s(n-2) \neq n-1$. Suppose $\sigma^s(n-2) \equiv n-1 \pmod{n-1} \Rightarrow s \equiv 1$, which is a contradiction. Suppose $\sigma^s(l-1) \equiv n-1 \Rightarrow s+l \equiv n \pmod{n-1}$. Subtracting two equations $s+2l \equiv 2$ and $s+l \equiv n \Rightarrow l \equiv 2-n \pmod{n-1}$, which is a contradiction. One of the following two sets has three distinct elements: $\{\sigma^s(m-1), n-2l, n-1\}$ or $\{\sigma^s(l-1), \sigma^s(m-1), n-1\}$.

Consider G_2 with $l \leq \frac{n}{2}-1$, $m = n-1$, $2 \leq i \leq n-1$, $j = n$.

1. G_2 with $l \leq \frac{n}{2}-2$.

$$\begin{pmatrix} \sigma^s(l-1) & \sigma^s(m-1) \\ n-2(l+1) & n-1 \end{pmatrix}$$

We wish to show there are three distinct entries: $\sigma^s(l-1) \neq \sigma^s(n-2)$, $\sigma^s(l-1) \neq n-2(l+1)$, $\sigma^s(n-2) \neq n-2(l+1)$. Suppose $n-2l \equiv \sigma^s(n-2) \Rightarrow n-2l \equiv s+n-2 \pmod{n-1} \Rightarrow s \equiv -2l+2 \pmod{n-1}$, which is a contradiction. Suppose $\sigma^s(n-2) \equiv n-2(l+1) \Rightarrow s \equiv -2l \pmod{n-1}$, which is a contradiction. Suppose $\sigma^s(l-1) \equiv n-2(l+1) \Rightarrow s+3l \equiv n-1 \pmod{n-1}$ and we wish to show there are three distinct entries: $\sigma^s(l-1) \neq \sigma^s(n-2)$, $\sigma^s(n-2) \neq n-1$, $\sigma^s(l-1) \neq n-1$. Suppose $\sigma^s(n-2) \equiv n-1 \pmod{n-1} \Rightarrow s \equiv 1 \Rightarrow i \equiv n$, which is a contradiction. Suppose $\sigma^s(l-1) \equiv n-1 \pmod{n-1} \Rightarrow s+l \equiv n$. Subtracting two equations $s+3l \equiv n-1$ and $s+l \equiv n \Rightarrow 2l \equiv -1 \pmod{n-1}$, which is a contradiction. One of

the following two sets has three distinct elements: $\{\sigma^s(l-1), \sigma^s(m-1), n-2(l+1)\}$ or $\{\sigma^s(l-1), \sigma^s(m-1), n-1\}$.

2. G_2 with $l = \frac{n}{2} - 1$.

$$\begin{pmatrix} \sigma^s(l-1) & \sigma^s(m-1) \\ Y & n-1 \end{pmatrix}$$

If K *is Even* $\Rightarrow Y = \frac{n}{2} - 2$. We wish to show there are three distinct entries: $\sigma^s(\frac{n}{2}-2) \neq \sigma^s(n-2)$, $\sigma^s(\frac{n}{2}-2) \neq \frac{n}{2}-2$, $\sigma^s(n-2) \neq \frac{n}{2}-2$. First two inequalities are correct. $\sigma^s(n-2) \equiv \frac{n}{2} - 2 \Rightarrow s + n - 2 \equiv \frac{n}{2} - 2 \pmod{n-1} \Rightarrow s \equiv -\frac{n}{2}$, which is a contradiction. There are three distinct elements $\{\sigma^s(l-1), \sigma^s(m-1), Y\}$

If K *is Odd* $\Rightarrow Y = \frac{n}{2} + 1$. We wish to show there are three distinct entries: $\sigma^s(n-2) \neq n-1$. $\sigma^s(n-2) \neq \frac{n}{2}+1$, $\frac{n}{2}+1 \neq n-1$. Suppose $\sigma^s(n-2) \equiv n-1 \Rightarrow s+n-2 \equiv n-1 \Rightarrow s \equiv 1 \pmod{n-1} \Rightarrow i \equiv n$, which is a contradiction. Suppose $\frac{n}{2}+1 = n-1 \Rightarrow n = 4$, which is a contradiction. Suppose $\sigma^s(n-2) \equiv \frac{n}{2}+1 \Rightarrow s+\frac{n}{2} \equiv 3 \pmod{n-1}$. Since $s = (n+1)-i \Rightarrow i \equiv 1.5n-4$, this is a contradiction. There are three distinct elements $\{\sigma^s(m-1), n-1, Y\}$.

Consider G_3 with $l \leq \frac{n}{2} - 1$, $m = n-1$, $2 \leq i \leq n-1$, $j = n$.

1. G_3 with $2 \leq l \leq \frac{n-4}{6}$.

$$\begin{pmatrix} \sigma^s(l-1) & \sigma^s(m-1) \\ n-2l & n-1 \end{pmatrix}$$

We wish to show there are three distinct entries: $\sigma^s(l-1) \neq \sigma^s(n-2)$, $\sigma^s(l-1) \neq n-1$, $\sigma^s(n-2) \neq n-1$. Suppose $\sigma^s(n-2) \equiv n-1 \Rightarrow s+n-2 \equiv n-1 \Rightarrow i \equiv n \pmod{n-1}$, which is a contradiction. Suppose $\sigma^s(l-1) \equiv n-1 \Rightarrow s+l \equiv n \pmod{n-1}$ and we wish to show there are three distinct entries: $\sigma^s(n-2) \neq n-1$, $n-2l \neq n-1$, $\sigma^s(n-2) \neq n-2l$. Suppose $n-2l = n-1 \Rightarrow l = \frac{1}{2}$, which is a contradiction. Suppose $\sigma^s(n-2) \equiv n-2l \Rightarrow s+2l = 2 \pmod{n-1}$. Subtracting two equations $s+2l = 2$ and $s+l \equiv n \Rightarrow l \equiv 2-n \pmod{n-1}$, which is a contradiction. One of the following two sets has three distinct elements: $\{\sigma^s(l-1), \sigma^s(m-1), n-1\}$ or $\{\sigma^s(m-1), n-2l, n-1\}$.

2. G_3 with $\frac{n-4}{6} \leq l \leq \frac{n}{2} - 2$.

$$\begin{pmatrix} \sigma^s(l-1) & \sigma^s(m-1) \\ n-2(l+1) & n-1 \end{pmatrix}$$

We wish to show there are three distinct entries: $\sigma^s(l-1) \neq \sigma^s(n-2)$, $\sigma^s(n-2) \neq n-1$, $\sigma^s(l-1) \neq n-1$. Suppose $\sigma^s(n-2) \equiv n-1 \Rightarrow i \equiv n \pmod{n-1}$, which is a contradiction. Suppose $\sigma^s(l-1) \equiv n-1 \Rightarrow s+l \equiv n \pmod{n-1}$ and we wish to show there are three distinct entries: $\sigma^s(l-1) \neq \sigma^s(n-2)$, $\sigma^s(l-1) \neq n-2(l+1)$, $\sigma^s(n-2) \neq n-2(l+1)$. Suppose $\sigma^s(n-2) \equiv n-2(l+1) \pmod{n-1} \Rightarrow s \equiv -2l \pmod{n-1}$, which is a contradiction. Suppose $\sigma^s(l-1) \equiv n-2(l+1) \Rightarrow s+3l \equiv n-1 \pmod{n-1}$. Subtracting this equation from $s+l \equiv n \pmod{n-1} \Rightarrow 2l \equiv -1$, which is a contradiction. One of the following two sets has three distinct elements: $\{\sigma^s(l-1), \sigma^s(m-1), n-1\}$ or $\{\sigma^s(m-1), \sigma^s(l-1), n-2(l+1)\}$.

3. G_3 with $l = \frac{n}{2} - 1$.

$$\begin{pmatrix} \sigma^s(l-1) & \sigma^s(m-1) \\ n-9 & n-1 \end{pmatrix}$$

We wish to show there are three distinct entries: $n-9 \neq n-1$, $\sigma^s(n-2) \neq n-1$, $\sigma^s(n-2) \neq n-9$. Suppose $\sigma^s(n-2) \equiv n-1 \Rightarrow s \equiv 1 \pmod{n-1}$, which is a contradiction. Suppose $\sigma^s(n-2) \equiv n-9 \Rightarrow s \equiv -7 \pmod{n-1}$, which is a contradiction. There are three distinct elements $\{\sigma^s(m-1), n-1, n-9\}$.

Exceptions: For $n = 16$ three distinct elements are: $\{\sigma^s(7), 5, 15\}$. For $n = 22$ three distinct elements are: $\{\sigma^s(10), 21, 17\}$.

Case 4: We take matrix $G(i,j;l,m)$ with $\frac{n}{2} \leq l \leq n-2$, ($m = n-1$ or $m = n$), $2 \leq i \leq n-1$, $j = n$. Three distinct entries are $\{\sigma^s(l-1), \sigma^s(m-1), n\}$.

Case 5: We take matrix $G(i,j;l,m)$ with $l = n-1$, $m = n$, $2 \leq i \leq n-1$, $j = n$. Three distinct entries are $\{n-1, \sigma^s(l-1), \sigma^s(m-1)\}$

Step 4:

For matrix $G(i,j;l,m)$ with $i = 1$, $2 \leq j \leq \mathbf{n}$, $l = 1$, $2 \leq m \leq n$ we look at four cases and consider every matrix type defined above of even $n \pmod 6$.

Case 1: We take matrix $G(i,j;l,m)$ with $l = 1$, $2 \leq m \leq n$, $i = 1$, $j = 2$.

$$\begin{pmatrix} 1 & m \\ 3 & m-1 \end{pmatrix}$$

We know that $m \neq 1$ and $m \neq m-1$. If $m-1 \neq 1 \Rightarrow$ we have three distinct entries. Suppose $m-1 = 1 \Rightarrow m = 2 \Rightarrow$ we have three distinct entries. One of the following two sets has three distinct elements: $\{m, 1, m-1\}$ or $\{1, 3, m\}$.

Case 2: We take matrix $G(i,j;l,m)$ with $l = 1$, $2 \leq m \leq n$, $i = 1$, $3 \leq j \leq \frac{n}{2}+1$.

$$\begin{pmatrix} 1 & m \\ n & X \end{pmatrix}$$

There are three distinct entries $\{1, m, n\}$.

Case 3: We take matrix $G(i,j;l,m)$ with $l = 1$, $2 \leq m \leq n$, $i = 1$, $\frac{n}{2}+2 \leq j \leq n-1$, and let $t = (n+1)-j \Rightarrow t = \frac{n}{2}-1$.

Consider G_1 and let $t = (n+1)-j$.

$$\begin{pmatrix} 1 & m \\ 2(j-1)-n & \sigma^t(m-1) \end{pmatrix}$$

We wish to show there are three distinct entries: $m \neq 1$, $m \neq \sigma^t(m-1)$, $1 \neq \sigma^t(m-1)$. Suppose $1 \equiv \sigma^{(t)}(m-1) \Rightarrow t \equiv 2-n$. Hence $t = (n+1)-j \Rightarrow n-1 \equiv j-m \pmod{n-1}$ and we wish to show there are three distinct entries: $m \neq 1$, $2j-2-n \neq 1$, $m \neq 2j-2-n$. Suppose $2j-2-n = 1 \Rightarrow j = \frac{n}{2} - \frac{1}{2}$. Since $j > \frac{n}{2}+1$, this is a contradiction. Suppose $m = 2j-2-n$, subtracting it from $n-1 \equiv j-m \Rightarrow j \equiv 3$, which is a contradiction. One of the following two sets has three distinct elements: $\{m, \sigma^t(m-1), 1\}$ or $\{m, 2j-2-n, 1\}$

Consider G_2.

1. G_2 with $j = \frac{n}{2}+2$.

$$\begin{pmatrix} 1 & m \\ Y & \sigma^t(m-1) \end{pmatrix}$$

If K *is Even* $\Rightarrow Y = \frac{n}{2} - 2$. We wish to show there are three distinct entries: $m \neq 1$, $\frac{n}{2} - 2 \neq 1$, $\frac{n}{2} - 2 \neq m$. Suppose $\frac{n}{2} - 2 = 1 \Rightarrow n = 6$, which is a contradiction. Suppose $\frac{n}{2} - 2 = m \Rightarrow n = 2m + 4$ and we wish to show there are three distinct entries: $\sigma^{\frac{n}{2}-1}(m-1) \neq m$, $m \neq 1$, $\sigma^{\frac{n}{2}-1}(m-1) \neq 1$. Suppose $\sigma^{\frac{n}{2}-1}(m-1) \equiv 1 \Rightarrow n \equiv 6 - 2m$. Subtracting it from e $n \equiv 6 - 2m \Rightarrow m \equiv \frac{1}{2} \pmod{n-1}$, which is a contradiction. One of the following two sets has three distinct elements: $\{m, Y, 1\}$ or $\{m, \sigma^t(m-1), 1\}$

If K *is Odd* $\Rightarrow Y = \frac{n}{2} + 1$. We wish to show there are three distinct entries: $m \neq 1$, $\frac{n}{2}+1 \neq 1$, $\frac{n}{2}+1 \neq m$. First two inequalities are correct. Suppose $\frac{n}{2}+1 = m \Rightarrow n = 2m - 2$ and we wish to show there are three distinct entries: $\sigma^{\frac{n}{2}-1}(m-1) \neq m$, $m \neq 1$, $\sigma^{\frac{n}{2}-1}(m-1) \neq 1$. Suppose $\sigma^{\frac{n}{2}-1}(m-1) \equiv 1 \Rightarrow n \equiv 2 - 2m \pmod{n-1}$. Subtracting it from $n = 2m - 2 \Rightarrow m \equiv 1$, which is a contradiction. One of the following two sets has three distinct elements: $\{m, Y, 1\}$ or $\{m, \sigma^t(m-1), 1\}$

2. G_2 with $\frac{n}{2} + 3 \leq j \leq n - 1$.

$$\begin{pmatrix} 1 & m \\ 2(j-2) - n & \sigma^t(m-1) \end{pmatrix}$$

We wish to show there are three distinct entries: $m \neq 1$, $m \neq \sigma^t(m-1)$, $\sigma^t(m-1) \neq 1$. Suppose $\sigma^t(m-1) \equiv 1 \Rightarrow j \equiv n - 1 + m \pmod{n-1}$. Hence $j \leq n-1$, this is a contradiction There are three distinct entries $\{m, \sigma^t(m-1), 1)\}$.

Consider G_3.

1. G_3 with $j = \frac{n}{2} + 2$.

$$\begin{pmatrix} 1 & m \\ n - 9 & \sigma^t(m-1) \end{pmatrix}$$

We wish to show there are three distinct entries: $m \neq 1$, $\sigma^t(m-1) \neq m$, $\sigma^t(m-1) \neq 1$. First two inequalities are correct. Suppose $\sigma^t(m-1) \equiv 1 \Rightarrow m \equiv 2 - \frac{n}{2} \pmod{n-1}$, which is a contradiction. Three distinct elements: $\{m, \sigma^t(m-1), 1\}$.

Exceptions: For $n = 16$ following the argument above we have that three distinct elements $\{m, \sigma^t(m-1), 1\}$. For $n = 22$ three distinct entries are $\{m, \sigma^t(m-1), 1\}$.

2. G_3 with $\frac{n}{2} + 3 \leq j \leq \frac{5n+4}{6}$.

$$\begin{pmatrix} 1 & m \\ 2(j-2) - n & \sigma^t(m-1) \end{pmatrix}$$

We wish to show there are three distinct entries: $m \neq 1$, $\sigma^t(m-1) \neq m$, $\sigma^t(m-1) \neq 1$. First two inequalities are correct. Suppose $\sigma^t(m-1) \equiv 1 \Rightarrow n - j + m \equiv 1 \pmod{n-1}$ and we wish to show there are three distinct entries: $m \neq 1$, $2(j-2)-n \neq 1$, $2(j-2)-n \neq m$. Suppose $2(j-2)-n = 1 \Rightarrow j = \frac{n+5}{2} \Rightarrow j = \frac{n}{2}+2.5$ and since $\frac{n}{2} + 3 \leq j$, this is a contradiction. Suppose $2(j-2) - n = m \Rightarrow 2j = n + m + 4$. Subtracting this equation from $n - j + m \equiv 1 \Rightarrow j \equiv 5 \pmod{n-1}$. Hence $n \geq 16$ and $j \leq \frac{5n+4}{6} \Rightarrow j \geq 14$, which is a contradiction. One of the following two sets has three distinct elements: $\{m, \sigma^t(m-1), 1\}$ or $\{m, 1, 2(j-2) - n\}$.

3. G_3 with $\frac{5n+10}{6} \leq j \leq n - 1$.

$$\begin{pmatrix} 1 & m \\ 2(j-1)-n & \sigma^t(m-1) \end{pmatrix}$$

We wish to show there are three distinct entries: $m \neq 1$, $\sigma^t(m-1) \neq m$, $\sigma^t(m-1) \neq 1$. First two inequalities are correct. Suppose $\sigma^t(m-1) \equiv 1 \Rightarrow n-j+m \equiv 1 \pmod{n-1}$ and we wish to show there are three distinct entries: $m \neq 1$, $2(j-1)-n \neq 1$, $2(j-1)-n \neq m$. Suppose $2(j-1)-n = 1 \Rightarrow j = \frac{n}{2}+1.5$ and since $\frac{5n+10}{6} \leq j$, this is a contradiction. Suppose $2(j-1)-n = m$, subtracting it from $m \equiv j+1-n \Rightarrow j \equiv 3$. Hence $n \geq 16$ and $j \geq \frac{5n+10}{6} \Rightarrow j \geq 15$, which is a contradiction. One of the following two sets has three distinct elements: $\{m, \sigma^t(m-1), 1\}$ or $\{m, 1, 2(j-1)-n\}$.

Case 4: We take matrix $G(i,j;l,m)$ with $l=1$, $2 \leq m \leq n$, $i=1$, $j=n$,
Consider G_1 with $2 \leq m \leq \frac{n}{2}-1$ and G_3 with $2 \leq m \leq \frac{n-4}{6}$

$$\begin{pmatrix} 1 & m \\ n-2 & n-2m \end{pmatrix}$$

We wish to show there are three distinct entries: $n-2 \neq 1$, $m \neq 1$, $m \neq n-2$. Suppose $m = n-2 \Rightarrow n = m+2$ and wish to show there are three distinct entries: $n-2 \neq n-2m$, $n-2 \neq 1$, $n-2m \neq 1$. First two inequalities are correct. Suppose $n-2m = 1$, subtracting it from $n = m+2 \Rightarrow m = 1$, which is a contradiction. One of the following two sets has three distinct elements: $\{m, n-2, 1\}$ or $\{n-2, 1, n-2m\}$.

Consider G_2 with $2 \leq m \leq \frac{n}{2}-1$ and G_3 with $\frac{n+2}{6} \leq m \leq \frac{n}{2}-1$.

1. G_2 with $2 \leq m \leq \frac{n}{2}-2$ and G_3 with $\frac{n+2}{6} \leq m \leq \frac{n}{2}-2$.

$$\begin{pmatrix} 1 & m \\ n-2 & n-2(m+1) \end{pmatrix}$$

We wish to show there are three distinct entries: $n-2 \neq 1$, $m \neq 1$, $m \neq n-2$. Assume $m = n-2$ and wish to show there are three distinct entries: $n-2 \neq n-2(m+1)$, $n-2 \neq 1$, $n-2(m+1) \neq 1$. Suppose $n-2(m+1) = 1$, subtracting it from $n = m+2 \Rightarrow m = -1$, which is a contradiction. One of the following two sets has three distinct elements: $\{m, n-2, 1\}$ or $\{n-2, 1, n-2(m+1)\}$.

2. G_2 and G_3 with $m = \frac{n}{2}-1$.

$$\begin{pmatrix} 1 & \frac{n}{2}-1 \\ n-2 & Y \end{pmatrix}$$

Three distinct elements are $\{1, n-2, \frac{n}{2}-1\}$.

Consider G_1, G_2, G_3

1. $G(i,j;l,m)$ with ($\frac{n}{2} \leq m \leq n-2$ or $m = n$). Three distinct entries are $\{1, n-2, n)\}$.

2. $G(i,j;l,m)$ with $m = n-1$. Three distinct entries are $\{1, n-1, n-2\}$.

Step 5:

For matrix $G(i,j;l,m)$ with $i=2$, $3 \leq j \leq \mathbf{n}$, $l=1$, $2 \leq m \leq n$ we look at three cases and consider every matrix type defined above of even $n \pmod 6$.

Case 1: We take the submatrix of $G(i,j;l,m)$ with $l=1$, $2 \leq m \leq n$, $i=2$, $3 \leq j \leq \frac{n}{2}+1$.

$$\begin{pmatrix} 3 & m-1 \\ n & \sigma^t(m-1) \end{pmatrix}$$

There are three distinct entries $\{m-1, \sigma^t(m-1), n\}$.

Case 2: We take the submatrix of $G(i,j;l,m)$ with $l=1$, $2 \le m \le n$, $i=2$, $\frac{n}{2}+2 \le j \le n-1$, and let $t=(n+1)-j$

Consider G_1

$$\begin{pmatrix} 3 & m-1 \\ 2(j-1)-n & \sigma^t(m-1) \end{pmatrix}$$

We wish to show there are three distinct entries: $m-1 \ne \sigma^t(m-1)$, $2j-n-2 \ne \sigma^t(m-1)$, $2j-n-2 \ne m-1$. The first inequality is correct. Suppose $2j-n-2 \equiv \sigma^{(t)}(m-1) \Rightarrow 3j \equiv 2n+2+m \pmod{n-1}$. Suppose $2j-n-2 = m-1 \Rightarrow 2j = n+1+m$ and we wish to show there are three distinct entries: $m-1 \ne 3$, $m-1 \ne \sigma^t(m-1)$, $\sigma^t(m-1) \ne 3$. Suppose $m-1=3 \Rightarrow m=4$. Suppose $\sigma^t(m-1) \equiv 3 \Rightarrow j \equiv n+m-3 \pmod{n-1}$. Subtracting two equations $m=4$ and $3j \equiv 2n+2+m \Rightarrow 3j \equiv 2n+6$ and hence $n=2+6k \Rightarrow 3j \equiv 2(6k+5)$, right part is not divisible by 3, which is a contradiction. Subtracting two equations $3j = 2n+2+m$ and $j \equiv n+m-3 \Rightarrow 2j \equiv n+5$, and since n is even, j is not integer, which is a contradiction. Subtracting two equations $m=4$ and $2j=n+1+m \Rightarrow 2j=n+5$, which is a contradiction. Subtracting two equations $2j=n+1+m$ and $j \equiv n+m-3 \Rightarrow j \equiv 4$ and since $j \ge \frac{n}{2}+2$, this is a contradiction. One of the following two sets has three distinct elements: $\{m-1, \sigma^t(m-1), 2j-n-2\}$ or $\{m-1, 3, \sigma^t(m-1)\}$.

Consider G_2

1. G_2 with $l=1$, $2 \le m \le n$, $i=2$, $j=\frac{n}{2}+2$, $t=(n+1)-j \Rightarrow t=\frac{n}{2}-1$.

$$\begin{pmatrix} 3 & m-1 \\ Y & \sigma^t(m-1) \end{pmatrix}$$

If K *is Even* $\Rightarrow Y = \frac{n}{2}-2$. We wish to show there are three distinct entries: $m-1 \ne \sigma^t(m-1)$, $\frac{n}{2}-2 \ne \sigma^t(m-1)$, $\frac{n}{2}-2 \ne m-1$. The first inequality is correct. Suppose $\frac{n}{2}-2 \equiv \sigma^t(m-1) \Rightarrow \frac{n}{2}-2 \equiv \frac{n}{2}-1+m-1 \Rightarrow m \equiv 0 \pmod{n-1}$, which is a contradiction. Suppose $\frac{n}{2}-2 = m-1 \Rightarrow \frac{n}{2}-1 = m$ and we wish to show there are three distinct entries: $m-1 \ne 3$, $m-1 \ne \sigma^t(m-1)$, $\sigma^t(m-1) \ne 3$. Suppose $\sigma^t(m-1) \equiv 3 \Rightarrow \frac{n}{2}-1+m-1 \equiv 3 \Rightarrow n-10 \equiv -2m \pmod{n-1} \Rightarrow n < 10$, which is a contradiction. Suppose $m-1=3 \Rightarrow m=4$ and since $\frac{n}{2}-1=m \Rightarrow n=10$, this is a contradiction. One of the following two sets has three distinct elements: $\{m-1, \sigma^t(m-1), Y\}$ or $\{m-1, 3, \sigma^t(m-1)\}$.

If K *is Odd* $\Rightarrow Y = \frac{n}{2}+1$, and let $t=(n+1)-j \Rightarrow t=\frac{n}{2}-1$. We wish to show there are three distinct entries: $m-1 \ne \sigma^t(m-1)$, $\frac{n}{2}+1 \ne \sigma^t(m-1)$, $\frac{n}{2}+1 \ne m-1$. The first inequality is correct. Suppose $\frac{n}{2}+1 \equiv \sigma^t(m-1) \Rightarrow m \equiv 3 \pmod{n-1}$. Suppose $\frac{n}{2}+1 = m-1 \Rightarrow n = 2m-4$ and we wish to show there are three distinct entries: $m-1 \ne 3$, $m-1 \ne \sigma^t(m-1)$, $\sigma^t(m-1) \ne 3$. From the argument above (for even k) we have $\sigma^t(m-1) \ne 3$ is correct. Suppose $m-1=3 \Rightarrow m=4$ and since $m \equiv 3$ we have a contradiction. Taking two equations $n=2m-4$ and

$m = 4 \Rightarrow n = 4$ we get a contradiction. One of the following two sets has three distinct elements: $\{m-1, \sigma^t(m-1), Y\}$ or $\{m-1, 3, \sigma^t(m-1)\}$.

2. G_2 with $l = 1$, $2 \le m \le n$, $i = 2$, $\frac{n}{2} + 3 \le j \le n - 1$, and let $t = (n+1) - j$.

$$\begin{pmatrix} 3 & m-1 \\ 2(j-2)-n & \sigma^t(m-1) \end{pmatrix}$$

We wish to show there are three distinct entries: $m - 1 \neq \sigma^t(m-1)$, $2(j-2) - n \neq \sigma^t(m-1)$, $2(j-2) - n \neq m - 1$. The first inequality is correct. Suppose $2(j-2) - n \equiv \sigma^t(m-1) \Rightarrow 3j - 4 - 2n \equiv m \pmod{n-1}$. Suppose $2(j-2) - n = m - 1 \Rightarrow 2j - n - 3 = m$ and we wish to show there are three distinct entries: $m - 1 \neq 3$, $m - 1 \neq \sigma^t(m-1)$, $\sigma^t(m-1) \neq 3$. Suppose $m - 1 = 3 \Rightarrow m = 4$ and since $3j - 4 - 2n \equiv m \Rightarrow 3j \equiv 2n + 8$ and since $n = 6 + 6k \Rightarrow j \equiv 4k + 6\frac{2}{3}$ which is a contradiction. Now take $2j - n - 3 = m$ and $m = 4 \Rightarrow j = \frac{n}{2} + 2.5$, which is a contradiction. Suppose $\sigma^t(m-1) \equiv 3 \Rightarrow m \equiv 3 + j - n \pmod{n-1}$ and since $3j - 4 - 2n \equiv m \Rightarrow 2j \equiv n + 7 \pmod{n-1} \Rightarrow j \equiv \frac{n}{2} + 3.5$, which is a contradiction. Now take $2j - n - 3 = m \Rightarrow j = 6$. Since $j \ge \frac{n}{2} + 3$, this is a contradiction. One of the following two sets has three distinct elements: $\{m-1, \sigma^t(m-1), 2(j-1) - n\}$ or $\{m-1, 3, \sigma^t(m-1)\}$.

Consider G_3

1. G_3 with $l = 1$, $2 \le m \le n$, $i = 2$, $j = \frac{n}{2} + 2$, and let $t = (n+1) - j \Rightarrow t = \frac{n}{2} - 1$.

$$\begin{pmatrix} 3 & m-1 \\ n-9 & \sigma^t(m-1) \end{pmatrix}$$

We wish to show there are three distinct entries: $m - 1 \neq \sigma^t(m-1)$, $n - 9 \neq \sigma^t(m-1)$, $n - 9 \neq m - 1$. The first inequality is correct. Suppose $n - 9 \equiv \sigma^t(m-1) \Rightarrow n \equiv 2m + 14$. Suppose $n - 9 = m - 1 \Rightarrow n = m + 8$. We wish to show there are three distinct entries: $m - 1 \neq \sigma^t(m-1)$, $m - 1 \neq 3$, $\sigma^t(m-1) \neq 3$. Suppose $m - 1 = 3 \Rightarrow m = 4$ and since $n \equiv 2m + 14 \Rightarrow n \equiv 22$, which is a contradiction. Now take $n = m + 8$ and since $m = 4 \Rightarrow n = 12$, which is a contradiction. Suppose $\sigma^t(m-1) \equiv 3 \Rightarrow n + 2m \equiv 10 \pmod{n-1}$ and since $n \equiv 2m + 14 \Rightarrow m \equiv -1$, which is a contradiction. Now take $n = m + 8$ and since $n + 2m \equiv 10 \Rightarrow m = \frac{2}{3}$, which is a contradiction. One of the following two sets has three distinct elements: $\{m-1, \sigma^t(m-1), n-9\}$ or $\{m-1, 3, \sigma^t(m-1)\}$.

Exceptions: For $n = 16$ following the argument above we have three distinct elements: $\{m-1, \sigma^t(m-1), 5\}$ or $\{m-1, 3, \sigma^t(m-1)\}$. For$n = 22$

$$\begin{pmatrix} 3 & m-1 \\ 17 & \sigma^{10}(m-1) \end{pmatrix}$$

We wish to show there are three distinct entries: $17 \neq 3$, $\sigma^{10}(m-1) \neq 3$, $\sigma^{10}(m-1) \neq 17$. Suppose $\sigma^{10}(m-1) \equiv 3 \Rightarrow m \equiv -6$, which is a contradiction. Suppose $\sigma^{10}(m-1) \equiv 17 \Rightarrow m \equiv 8$. We wish to show there are three distinct entries: $17 \neq 3$, $m - 1 \neq 3$, $m - 1 \neq 17$. Suppose $m - 1 = 3$. Since $m \equiv 8$, this is a contradiction. Suppose $m - 1 = 17$. Since $m \equiv 8$, this is a contradiction. One of the following two sets has three distinct elements: $\{17, \sigma^t(m-1), 3\}$ or $\{3, 17, m-1\}$.

2. G_3 with $l = 1$, $2 \le m \le n$, $i = 2$, $\frac{n}{2} + 3 \le j \le \frac{5n+4}{6}$, and let $t = (n+1) - j$

$$\begin{pmatrix} 3 & m-1 \\ 2(j-2)-n & \sigma^t(m-1) \end{pmatrix}$$

We wish to show there are three distinct entries: $m-1 \neq \sigma^t(m-1)$, $2j-4-n \neq \sigma^t(m-1)$, $2j-4-n \neq m-1$. Suppose $2j-4-n \equiv \sigma^t(m-1) \Rightarrow 2n+m+4 \equiv 3j \pmod{n-1}$. Suppose $2j-4-n = m-1 \Rightarrow n+m+3 = 2j$. We wish to show there are three distinct entries: $m-1 \neq 3$, $m-1 \neq \sigma^t(m-1)$, $\sigma^t(m-1) \neq 3$. Suppose $m-1 = 3 \Rightarrow m = 4$. Hence $2n+m+4 \equiv 3j$ and $m = 4 \Rightarrow 2n+8 \equiv 3j \pmod{n-1}$. Since $n = 4+6k \Rightarrow j \equiv 4k+5\frac{1}{3}$, which is a contradiction. Hence $n+m+3 = 2j$ and $m = 4 \Rightarrow j \equiv \frac{n}{2}+3.5$, which is a contradiction. Suppose $\sigma^t(m-1) \equiv 3 \Rightarrow j \equiv n+m-3 \pmod{n-1}$. Hence $2n+m+4 \equiv 3j$ and $j \equiv n+m-3$, subtracting them $\Rightarrow j \equiv \frac{n}{2}+3.5$, which is a contradiction. Hence $n+m+3 = 2j$ and $j \equiv n+m-3$, subtracting them $\Rightarrow j \equiv 6 \pmod{n-1}$. Since $j \geq \frac{n}{2}+3$, which is a contradiction. One of the following two sets has three distinct elements: $\{m-1, \sigma^t(m-1), 2j-4-n\}$ or $\{m-1, 3, \sigma^t(m-1)\}$.

3. G_3 with $l = 1$, $2 \leq m \leq n$, $i = 2$, $\frac{5n+10}{6} \leq j \leq n-1$, and let $t = (n+1)-j$.

$$\begin{pmatrix} 3 & m-1 \\ 2(j-1)-n & \sigma^t(m-1) \end{pmatrix}$$

We wish to show there are three distinct entries: $m-1 \neq \sigma^t(m-1)$, $2j-2-n \neq \sigma^t(m-1)$, $2j-2-n \neq m-1$. Suppose $2j-2-n \equiv \sigma^t(m-1) \Rightarrow 2n+m+2 \equiv 3j \pmod{n-1}$. Suppose $2j-2-n = m-1 \Rightarrow n+m+1 = 2j$. We wish to show there are three distinct entries: $m-1 \neq 3$, $m-1 \neq \sigma^t(m-1)$, $\sigma^t(m-1) \neq 3$. Suppose $m-1 = 3 \Rightarrow m = 4$. Subtracting two equations $2n+m+2 \equiv 3j$ and $m = 4 \Rightarrow 2n+6 \equiv 3j \pmod{n-1}$. Since $j \geq \frac{5n+10}{6}$, this is a contradiction. Subtracting two equations $n+m+1 = 2j$ and $m = 4 \Rightarrow j = \frac{n}{2}+2.5$, which is a contradiction. Suppose $\sigma^t(m-1) \equiv 3 \Rightarrow j \equiv n+m-3 \pmod{n-1}$. Subtracting two equations $2n+m+2 \equiv 3j$ and $j \equiv n+m-3 \Rightarrow j = \frac{n}{2}+2.5$, which is a contradiction. Subtracting two equations $n+m+1 = 2j$ and $j \equiv n+m-3 \Rightarrow j \equiv 4$. Since $j \geq \frac{5n+10}{6}$ and $n \geq 28$, this is a contradiction. One of the following two sets has three distinct elements: $\{m-1, \sigma^t(m-1), 2j-2-n\}$ or $\{m-1, 3, \sigma^t(m-1)\}$.

Case 3: We take matrix $G(i,j;l,m)$ with $l = 1$, $2 \leq m \leq n$, $i = 2$, $j = n$.

Consider G_1 with $2 \leq m \leq \frac{n}{2}-1$ and *consider* G_3 with $2 \leq m \leq \frac{n-4}{6}$.

$$\begin{pmatrix} 3 & m-1 \\ n-2 & n-2m \end{pmatrix}$$

We wish to show there are three distinct entries: $n-2 \neq 3$, $m-1 \neq n-2$, $m-1 \neq 3$. Suppose $m-1 = 3 \Rightarrow m = 4$. We wish to show there are three distinct entries: $n-2 \neq n-2m$, $n-2 \neq 3$, $n-2m \neq 3$. Since $n-2m = 3$ and since $m = 4 \Rightarrow n = 11$, this is a contradiction. Suppose $m-1 = n-2 \Rightarrow m = n-1$ and since $n-2m = 3 \Rightarrow m = -2$, this is a contradiction. One of the following two sets has three distinct elements: $\{m-1, n-2, 3\}$ or $\{n-2, 3, n-2m\}$.

Consider G_2 with $2 \leq m \leq \frac{n}{2}-1$ and *consider* G_3 with $\frac{n+2}{6} \leq m \leq \frac{n}{2}-1$.

1. G_2 with $2 \leq m \leq \frac{n}{2}-2$ and G_3 with $\frac{n+2}{6} \leq m \leq \frac{n}{2}-2$, $l = 1$, $i = 2$, $j = n$.

$$\begin{pmatrix} 3 & m-1 \\ n-2 & n-2(m+1) \end{pmatrix}$$

We wish to show there are three distinct entries: $n-2 \neq 3$, $m-1 \neq n-2$, $m-1 \neq 3$. Suppose $m-1=3 \Rightarrow m=4$. We wish to show there are three distinct entries: $n-2 \neq n-2(m+1)$, $n-2 \neq 3$, $n-2(m+1) \neq 3$ Suppose $n-2(m+1)=3$ and since $m=4 \Rightarrow n=13$, which is a contradiction. One of the following two sets has three distinct elements: $\{m-1, n-2, 3\}$ or $\{n-2, 3, n-2(m+1)\}$.

2. G_2 and G_3 with $m=\frac{n}{2}-1$, $l=1$, $i=2$, $j=n$ Three distinct elements are $\{3, n-2, \frac{n}{2}-1\}$

Consider G_1,G_2 *and* G_3

1. $G(i,j;l,m)$ with $\frac{n}{2} \leq m \leq n-2$, $l=1$, $i=2$, $j=n$. Three distinct elements are $\{3, n-2, n\}$.

2. $G(i,j;l,m)$ with $m=n-1$, $l=1$, $i=2$, $j=n$. Three distinct elements are $\{3, n-1, n-2\}$.

3. $G(i,j;l,m)$ with $m=n$, $l=1$, $i=2$, $j=n$. Three distinct elements are $\{3, n-2, 1\}$.

Step 6:

For matrix $G(i,j;l,m)$ with $3 \leq i < j \leq \mathbf{n}-1$, $l=1$, $2 \leq m \leq n$ we look at two cases and consider every matrix type defined above of even $n \pmod 6$.

Case 1: We take the submatrix of $G(i,j;l,m)$ with $l=1$, $2 \leq m \leq n$, $3 \leq i \leq \frac{n}{2}+1$, $i \leq j \leq n-1$.

$$\begin{pmatrix} n & \sigma^s(m-1) \\ X & \sigma^t(m-1) \end{pmatrix}$$

There are three distinct entries $\{n, \sigma^s(m-1), \sigma^t(m-1)\}$.

Case 2: We take matrix $G(i,j;l,m)$ with $l=1$, $2 \leq m \leq n$, $\frac{n}{2}+2 \leq i < j \leq n-1$, and let $s=(n+1)-i$, $t=(n+1)-j$.

Consider G_1

$$\begin{pmatrix} 2(i-1)-n & \sigma^s(m-1) \\ 2(j-1)-n & \sigma^t(m-1) \end{pmatrix}$$

We wish to show there are three distinct entries: $\sigma^s(m-1) \neq \sigma^t(m-1)$, $2i-2-n \neq \sigma^s(m-1)$, $2i-2-n \neq \sigma^t(m-1)$. Suppose $2i-2-n \equiv \sigma^s(m-1) \Rightarrow 3i \equiv 2n+m+2 \pmod{n-1}$. Suppose $2i-2-n \equiv \sigma^t(m-1) \Rightarrow 2i+j \equiv 2n+m+2 \pmod{n-1}$. We wish to show there are three distinct entries: $\sigma^s(m-1) \neq \sigma^t(m-1)$, $2j-2-n \neq \sigma^t(m-1)$, $2j-2-n \neq \sigma^s(m-1)$. Suppose $2j-2-n \equiv \sigma^t(m-1) \Rightarrow 3j \equiv 2n+m+2$. Subtracting this equation from $3i \equiv 2n+m+2 \Rightarrow j \equiv i$, which is a contradiction. Subtracting the same equation from $2i+j \equiv 2n+m+2 \Rightarrow j \equiv i$, which is a contradiction. Suppose $2j-2-n \equiv \sigma^s(m-1) \Rightarrow 2j+i \equiv 2n+m+2$. Subtracting this equation from $3i \equiv 2n+m+2 \Rightarrow j \equiv i$, which is a contradiction. Subtracting the same equation from $2i+j \equiv 2n+m+2 \Rightarrow j \equiv i$, which is a contradiction. One of the following two sets has three distinct elements: $\{\sigma^s(m-1), \sigma^t(m-1), 2i-2-n\}$ or $\{\sigma^s(m-1), \sigma^t(m-1), 2j-2-n\}$.

Consider G_2 with $l = 1$, $2 \le m \le n$, $\frac{n}{2} + 2 \le i < j \le n - 1$

1. G_2 with $i = \frac{n}{2} + 2$, $\frac{n}{2} + 3 \le j \le n - 1$, and let $s = \frac{n}{2} - 1$, $t = (n + 1) - j$.

$$\begin{pmatrix} Y & \sigma^s(m-1) \\ 2(j-2)-n & \sigma^t(m-1) \end{pmatrix}$$

If K *is Even* $\Rightarrow Y = \frac{n}{2} - 2$. We wish to show there are three distinct entries: $\sigma^s(m-1) \neq \sigma^t(m-1)$, $\frac{n}{2} - 2 \neq \sigma^t(m-1)$ and $\frac{n}{2} - 2 \neq \sigma^s(m-1)$. Suppose $\frac{n}{2} - 2 \equiv \sigma^s(m-1) \Rightarrow m \equiv 0 \pmod{n-1}$, which is a contradiction. Suppose $\frac{n}{2} - 2 \equiv \sigma^t(m-1) \Rightarrow 2j \equiv \mathbf{n} + 2m + 4 \pmod{n-1}$ and we wish to show there are three distinct entries: $2(j-2) - n \neq \frac{n}{2} - 2$, $\frac{n}{2} - 2 \neq \sigma^s(m-1)$, $2(j-2) - n \neq \sigma^s(m-1)$. Suppose $\frac{n}{2} - 2 \equiv 2(j-2) - n \Rightarrow 4j \equiv 3n + 4$. For this type $n = 6 + 6k \Rightarrow j \equiv 4.5k + 5.5$, and since k is even j is not integer, which is a contradiction. Suppose $2(j-2) - n \equiv \sigma^s(m-1) \Rightarrow 4j \equiv 3n + 2m + 4 \pmod{n-1}$. Subtracting it from $2j \equiv \mathbf{n} + 2m + 4 \Rightarrow j \equiv n \pmod{n-1}$, which is a contradiction. One of the following two sets has three distinct elements: $\{\sigma^s(m-1), \sigma^t(m-1), \frac{n}{2} - 2\}$ or $\{\sigma^s(m-1), \frac{n}{2} - 2, 2j - 4 - n\}$.

If K *is Odd* $\Rightarrow Y = \frac{n}{2} + 1$. We wish to show there are three distinct entries: $Y \neq \sigma^s(m-1)$, $Y \neq 2j - n - 4$, $2j - n - 4 \neq \sigma^t(m-1)$. Suppose $\frac{n}{2} + 1 = 2j - n - 4 \Rightarrow 4j = 3n + 10 \Rightarrow j = 4.5k + 7$. Since k is odd, j is not integer, which is a contradiction. Suppose $\frac{n}{2} + 1 \equiv \sigma^t(m-1) \Rightarrow 2j \equiv n + 2m - 2 \pmod{n-1}$. Suppose $2j - n - 4 \equiv \sigma^t(m-1) \Rightarrow 3j \equiv 2n + 4 + m \pmod{n-1}$ and we wish to show there are three distinct entries: $\frac{n}{2} + 1 \neq 2j - n - 4$, $\frac{n}{2} + 1 \neq \sigma^s(m-1)$, $2j - n - 4 \neq \sigma^s(m-1)$. Suppose $\frac{n}{2} + 1 \equiv \sigma^s(m-1) \Rightarrow m \equiv 3 \pmod{n-1}$. Subtracting this equation from $2j \equiv n + 2m - 2 \Rightarrow j \equiv \frac{n}{2} + 2 \pmod{n-1}$ and since $\frac{n}{2} + 3 \le j$ this is a contradiction. Replacing $m \equiv 3$ in $3j \equiv 2n + 4 + m \Rightarrow 3j \equiv 12k + 22$, since right side is not divisible by 3 this is a contradiction. Suppose $2j - n - 4 \equiv \sigma^s(m-1) \Rightarrow 4j \equiv 3n + 2m + 4 \pmod{n-1}$. Subtracting it from $3j \equiv n + m + 4 \Rightarrow j \equiv n + m \Rightarrow j > n$, which is a contradiction. Subtracting it from $2j \equiv n + 2m - 2 \Rightarrow j \equiv n + 2m + 6 \Rightarrow j > n$, which is a contradiction One of the following two sets has three distinct elements: $\{\sigma^s(m-1), Y, 2j - 4 - n\}$ or $\{Y, \sigma^t(m-1), 2j - 4 - n\}$.

2. G_2 with $\frac{n}{2} + 3 \le i < j \le n - 1$ $l = 1$, $2 \le m \le n$, and let $s = (n + 1) - i$, $t = (n + 1) - j$.

$$\begin{pmatrix} 2(i-2)-n & \sigma^s(m-1) \\ 2(j-2)-n & \sigma^t(m-1) \end{pmatrix}$$

We wish to show there are three distinct entries: $\sigma^s(m-1) \neq \sigma^t(m-1)$, $2i - 4 - n \neq \sigma^s(m-1)$, $2i - 4 - n \neq \sigma^t(m-1)$. Suppose $2i - 4 - n \equiv \sigma^s(m-1) \Rightarrow 3i \equiv 2n + m + 4 \pmod{n-1}$. Suppose $2i - 4 - n \equiv \sigma^t(m-1) \Rightarrow 2i + j \equiv 2n + m + 4 \pmod{n-1}$. We wish to show there are three distinct entries: $\sigma^s(m-1) \neq \sigma^t(m-1)$, $2j - 4 - n \neq \sigma^t(m-1)$, $2j - 4 - n \neq \sigma^s(m-1)$. Suppose $2j - 4 - n \equiv \sigma^t(m-1) \Rightarrow 3j \equiv 2n + m + 4 \pmod{n-1}$. Subtracting this equation from $3i \equiv 2n + m + 4 \Rightarrow j \equiv i$, which is a contradiction. Subtracting the same equation from $2i + j \equiv 2n + m + 4 \Rightarrow j \equiv i$, which is a contradiction. Suppose $2j - 4 - n \equiv \sigma^s(m-1) \Rightarrow 2j + i \equiv 2n + m + 4 \pmod{n-1}$. Subtracting this equation from $3i \equiv 2n + m + 4 \Rightarrow j \equiv i$, which is a contradiction. Subtracting the same equation from $2i + j \equiv 2n + m + 4 \Rightarrow j \equiv i$, which is a contradiction. One of the following two sets has three distinct elements: $\{\sigma^s(m-1), \sigma^t(m-1), 2i - 4 - n\}$ or $\{\sigma^s(m-1), \sigma^t(m-1), 2j - 4 - n\}$.

Consider G_3 with $l = 1$, $2 \le m \le n$, $\frac{n}{2} + 2 \le i < j \le n - 1$

1. G_3 with $i = \frac{n}{2} + 2$, $\frac{n}{2} + 3 \le j \le \frac{5n+4}{6}$, $l = 1$, $2 \le m \le n$, and let $s = \frac{n}{2} - 1$, $t = (n+1) - j$.

$$\begin{pmatrix} n-9 & \sigma^s(m-1) \\ 2j-4-n & \sigma^t(m-1) \end{pmatrix}$$

We wish to show there are three distinct entries: $\sigma^s(m-1) \ne \sigma^t(m-1)$, $2j-4-n \ne \sigma^t(m-1)$, $2j-4-n \ne \sigma^s(m-1)$. Suppose $2j-4-n \equiv \sigma^s(m-1) \Rightarrow 4j \equiv 3n+2m+4 \pmod{n-1}$. Suppose $2j-4-n \equiv \sigma^t(m-1) \Rightarrow 3j \equiv 2n+m+4 \pmod{n-1}$. We wish to show there are three distinct entries: $\sigma^s(m-1) \ne \sigma^t(m-1)$, $n-9 \ne \sigma^s(m-1)$, $n-9 \ne \sigma^t(m-1)$. Suppose $n-9 \equiv \sigma^s(m-1) \Rightarrow n \equiv 2m+14 \pmod{n-1}$. Subtracting this equation from $4j \equiv 3n+2m+4 \Rightarrow j \equiv \frac{n}{2}+m+4.5 \Rightarrow j$ is not integer, which is a contradiction. Subtracting the same equation from $3j \equiv 2n+m+4 \Rightarrow n \equiv 3j-3m-18$. Since $n = 4+6k \Rightarrow j = 2k+m+7(\frac{1}{3}) \Rightarrow j$ is not integer, which is a contradiction. Suppose $n-9 \equiv \sigma^t(m-1) \Rightarrow j \equiv m+9 \pmod{n-1}$. Subtracting this equation from $4j \equiv 3n+2m+4 \Rightarrow j \equiv n+\frac{m-5}{3} \Rightarrow j > n$, which is a contradiction. Subtracting the same equation from $3j \equiv 2n+m+4 \Rightarrow n \equiv j+2.5$, which is a contradiction. One of the following two sets has three distinct elements: $\{\sigma^s(m-1), \sigma^t(m-1), 2j-4-n\}$ or $\{\sigma^s(m-1), \sigma^t(m-1), n-9\}$.

Exceptions: For $n = 16$ following the argument above we have three distinct elements: $\{\sigma^s(m-1), \sigma^t(m-1), 2j-4-n\}$ or $\{\sigma^s(m-1), \sigma^t(m-1), n-11\}$.

For $n = 22$

$$\begin{pmatrix} n-5 & \sigma^s(m-1) \\ 2j-4-n & \sigma^t(m-1) \end{pmatrix}$$

We wish to show there are three distinct entries: $\sigma^s(m-1) \ne \sigma^t(m-1)$, $2j-4-n \ne \sigma^t(m-1)$, $2j-4-n \ne \sigma^t(m-1)$. Suppose $2j-26 \equiv \sigma^{10}(m-1) \Rightarrow 2j \equiv 35+m \pmod{n-1}$. Suppose $2j-26 \equiv \sigma^t(m-1) \Rightarrow 3j \equiv 48+m \pmod{n-1}$. We wish to show there are three distinct entries: $\sigma^{10}(m-1) \ne \sigma^t(m-1)$, $17 \ne \sigma^{10}(m-1)$, $17 \ne \sigma^t(m-1)$. Suppose $17 \equiv \sigma^{10}(m-1) \Rightarrow m \equiv 8 \pmod{n-1}$. Suppose $17 \equiv \sigma^t(m-1) \Rightarrow j \equiv 5+m \pmod{n-1}$. Subtracting two equations $m \equiv 8$ and $2j \equiv 35+m \Rightarrow j \equiv 21.5$, which is a contradiction. Subtracting two equations $m \equiv 8$ and $3j \equiv 48+m \Rightarrow j \equiv 17\frac{2}{3}$, which is a contradiction. Subtracting two equations $j \equiv 5+m$ and $2j \equiv 35+m \Rightarrow j \equiv 30$ and since $n = 22$, this is a contradiction. Subtracting two equations $j \equiv 5+m$ and $3j \equiv 48+m \Rightarrow j \equiv 21.5$, which is a contradiction. One of the following two sets has three distinct elements: $\{\sigma^s(m-1), \sigma^t(m-1), 2j-4-n\}$ or $\{\sigma^s(m-1), \sigma^t(m-1), n-5\}$.

2. G_3 with $i = \frac{n}{2}+2$, $\frac{5n+10}{6} \le j \le n-1$, $l = 1$, $2 \le m \le n$, and let $t = (n+1)-j$, $s = \frac{n}{2} - 1$

$$\begin{pmatrix} n-9 & \sigma^s(m-1) \\ 2j-2-n & \sigma^t(m-1) \end{pmatrix}$$

We wish to show there are three distinct entries: $\sigma^s(m-1) \ne \sigma^t(m-1)$, $n-9 \ne \sigma^s(m-1)$, $n-9 \ne \sigma^t(m-1)$. Suppose $n-9 \equiv \sigma^s(m-1) \Rightarrow \frac{n}{2}-7 \equiv m \Rightarrow n \equiv 2m+14 \pmod{n-1}$. Suppose $n-9 \equiv \sigma^t(m-1) \Rightarrow j \equiv m+9 \pmod{n-1}$. We wish to show there are three distinct entries: $\sigma^s(m-1) \ne \sigma^t(m-1)$, $2j-2-n \ne \sigma^t(m-1)$, $2j-2-n \ne \sigma^s(m-1)$. Suppose $2j-2-n \equiv \sigma^s(m-1) \Rightarrow 4j \equiv 3n+2m \pmod{n-1}$.

Replacing $2m \equiv n-14 \Rightarrow j \equiv n-3.5$, which is a contradiction. Replacing $m \equiv j-9$ in $4j \equiv 3n+2m \Rightarrow j \equiv 1.5n-9 \Rightarrow j > n$, which is a contradiction. Suppose $2j-2-n \equiv \sigma^t(m-1) \Rightarrow 3j \equiv 2n+2+m \pmod{n-1}$. Replacing $m \equiv \frac{n}{2}-7$ in the equation $3j \equiv 2n+2+m \Rightarrow j \equiv \frac{5n-10}{6}$ and since $j \geq \frac{5n+10}{6}$ this is a contradiction. Subtracting the equation $3j \equiv 2n+2+m$ from $j \equiv m+9 \Rightarrow n \equiv j-3.5$, which is a contradiction. One of the following two sets has three distinct elements: $\{\sigma^s(m-1), \sigma^t(m-1), 2j-2-n\}$ or $\{\sigma^s(m-1), \sigma^t(m-1), n-9\}$

Exceptions: For $n=16$ following the argument above we have three distinct elements: $\{\sigma^s(m-1), \sigma^t(m-1), 2j-2-n\}$ or $\{\sigma^s(m-1), \sigma^t(m-1), n-11\}$. For $n=22$ following the argument above we have three distinct elements: $\{\sigma^s(m-1), \sigma^t(m-1), 2j-2-n\}$ or $\{\sigma^s(m-1), \sigma^t(m-1), n-5\}$.

3. G_3 with $\frac{n}{2}+3 \leq i < j \leq \frac{5n+4}{6}$, $l=1$, $2 \leq m \leq n$, and let $t=(n+1)-j$, $s=(n+1)-i$.

$$\begin{pmatrix} 2(i-2)-n & \sigma^s(m-1) \\ 2(j-2)-n & \sigma^t(m-1) \end{pmatrix}$$

This case is identical to G_2 **2.**. One of the following two sets has three distinct elements: $\{\sigma^s(m-1), \sigma^t(m-1), 2i-4-n\}$ or $\{\sigma^s(m-1), \sigma^t(m-1), 2j-4-n\}$.

4. G_3 with $\frac{5n+10}{6} < i < j \leq n-1$, $l=1$, $2 \leq m \leq n$, and let $s=(n+1)-i$, $t=(n+1)-j$.

$$\begin{pmatrix} 2(i-1)-n & \sigma^s(m-1) \\ 2(j-1)-n & \sigma^t(m-1) \end{pmatrix}$$

This case is identical to G_1. One of the following two sets has three distinct elements: $\{\sigma^s(m-1), \sigma^t(m-1), 2i-2-n\}$ or $\{\sigma^s(m-1), \sigma^t(m-1), 2j-2-n\}$.

5. G_3 with $\frac{n}{2}+3 \leq i \leq \frac{5n+4}{6}$, $\frac{5n+10}{6} \leq j \leq n-1$, $l=1$, $2 \leq m \leq n$, and let $s=(n+1)-i$, $t=(n+1)-j$

$$\begin{pmatrix} 2(i-2)-n & \sigma^s(m-1) \\ 2(j-1)-n & \sigma^t(m-1) \end{pmatrix}$$

We wish to show three are three distinct entries: $2(i-2)-n \neq 2(j-1)-n$, $2(i-2)-n \neq \sigma^s(m-1)$, $2(j-1)-n \neq \sigma^s(m-1)$. Suppose $2i-4-n \equiv \sigma^s(m-1) \Rightarrow 3i \equiv 2n+m+4 \pmod{n-1}$. Suppose $2j-2-n \equiv \sigma^s(m-1) \Rightarrow 2j+i \equiv 2n+m+2 \pmod{n-1}$. We wish to show three are three distinct entries: $2i-4-n \neq 2j-2-n$, $2j-2-n \neq \sigma^t(m-1)$, $2i-4-n \neq \sigma^t(m-1)$. Suppose $2j-2-n \equiv \sigma^t(m-1) \Rightarrow 3j \equiv 2n+m+2 \pmod{n-1}$. Subtracting this equation from $3i \equiv 2n+m+4 \Rightarrow j \equiv i-\frac{2}{3}$, which is a contradiction. Subtracting the same equation from $2i+j \equiv 2n+m+2 \Rightarrow j \equiv i$, which is a contradiction. Suppose $2i-4-n \equiv \sigma^t(m-1) \Rightarrow j+2i \equiv 2n+m+4 \pmod{n-1}$. Subtracting this equation from $3i \equiv 2n+m+4 \Rightarrow j \equiv i$, which is a contradiction. Subtracting the same equation from $2j+i \equiv 2n+m+2 \Rightarrow i \equiv j+2$, which is a contradiction. One of the following two sets has three distinct elements: $\{\sigma^s(m-1), 2(i-2)-n, 2(j-1)-n\}$ or $\{\sigma^t(m-1), 2(i-2)-n, 2(j-1)-n\}$.

Step 7:

For matrix $G(i,j;l,m)$ with $l=1$, $2 \leq m \leq n$, $3 \leq i \leq \frac{n}{2}+1$, $j=n$, and let $s=n+1-i$ and we look at four cases and consider every matrix type defined above of even $n \pmod 6$.

Case 1: We take matrix $G(i,j;l,m)$ with $l=1$, $2 \le m \le \frac{n}{2}-1$, $3 \le i \le \frac{n}{2}+1$, $j=n$, and let $s=n+1-i$.

Consider G_1 with $2 \le m \le \frac{n}{2}-1$ and G_3 with $2 \le m \le \frac{n-4}{6}$.

$$\begin{pmatrix} n & \sigma^s(m-1) \\ n-2 & n-2m \end{pmatrix}$$

There are three distinct elements $\{n, n-2, n-2m\}$

Consider G_2 with $2 \le m \le \frac{n}{2}-1$ and G_3 with $\frac{n+2}{6} \le m \le \frac{n}{2}-1$.
1. G_2 with $2 \le m \le \frac{n}{2}-2$ and G_3 with $\frac{n+2}{6} \le m \le \frac{n}{2}-2$.

$$\begin{pmatrix} n & \sigma^s(m-1) \\ n-2 & n-2(m+1) \end{pmatrix}$$

There are three distinct elements $\{n, n-2, n-2(m+1)\}$
2. G_2 and G_3 with $m = \frac{n}{2}-1$.

$$\begin{pmatrix} n & \sigma^s(m-1) \\ n-2 & Y \end{pmatrix}$$

For G_2: If K *is Even* $\Rightarrow Y = \frac{n}{2}-2$, there are three distinct entries $(n, n-2, \frac{n}{2}-2)$. If K *is Odd* $\Rightarrow Y = \frac{n}{2}+1$, there are three distinct entries $(n, n-2, \frac{n}{2}+1)$.

For G_3: There are three distinct entries $\{n, n-2, n-9\}$. Exceptions: For $n=16$ three distinct elements are $\{16, 14, 5\}$. For $n=22$ then there are three distinct entries $\{22, 20, 17\}$.

Case 2: We take matrix $G(i,j;l,m)$ with $l=1$, $j=n$, $3 \le i \le \frac{n}{2}+1$, $\frac{n}{2} \le m \le n-2$, and let $s=n+1-i$.

$$\begin{pmatrix} n & \sigma^s(m-1) \\ n-2 & n \end{pmatrix}$$

Suppose $\sigma^s(m-1) \equiv n-2 \Rightarrow m+2 \equiv i$, since the conditions for m and i this is a contradiction. There are three distinct elements $\{n, n-2, \sigma^s(m-1)\}$.

Case 3: We take matrix $G(i,j;l,m)$ with $3 \le i \le \frac{n}{2}+1$, $l=1$, $j=n$, $m=n-1$, and let $s=n+1-i$.

$$\begin{pmatrix} n & X \\ n-2 & n-1 \end{pmatrix}$$

Three distinct elements are $\{n, n-1, n-2\}$.

Case 4: We take matrix $G(i,j;l,m)$ with $l=1$, $j=n$, $3 \le i \le \frac{n}{2}+1$, $m=n$, and let $s=n+1-i$.

$$\begin{pmatrix} n & X \\ n-2 & 1 \end{pmatrix}$$

Three distinct elements are $\{n, n-2, 1\}$.

Step 8:

For matrix $G(i,j;l,m)$ with $l=1$, $2 \le m \le n$, $\frac{n}{2}+2 \le i \le n-1$, $j=n$, and let $s=n+1-i$ and we look at four cases and consider every matrix type defined above of even $n \pmod 6$.

Case 1: We take matrix $G(i,j;l,m)$ with $l=1$, $2 \le m \le \frac{n}{2}-1$, $\frac{n}{2}+2 \le i \le n-1$, $j=n$, and let $s=(n+1)-i, t=(n+1)-j$.

Consider G_1.

$$\begin{pmatrix} 2(i-1)-n & \sigma^s(m-1) \\ n-2 & n-2m \end{pmatrix}$$

We wish to show there are three distinct entries: $n-2 \neq n-2m$, $2i-2-n \neq n-2$, $2i-2-n \neq n-2m$. Suppose $2i-2-n = n-2m \Rightarrow i = n+1+m$, which is a contradiction. Three distinct elements are $\{n-2, n-2m, 2i-2-n\}$.

Consider G_2.

1. G_2 with $2 \le m \le \frac{n}{2}-2$, $i=\frac{n}{2}+2$.

$$\begin{pmatrix} Y & \sigma^s(m-1) \\ n-2 & n-2(m+1) \end{pmatrix}$$

If K *is Even* $\Rightarrow Y=\frac{n}{2}-2 \Rightarrow$, three distinct entries $\{n-2, n-2(m+1), Y\}$. If K *is Odd* $\Rightarrow Y=\frac{n}{2}+1 \Rightarrow$, three distinct entries $\{n-2, n-2(m+1), Y\}$.

2. G_2 with $2 \le m \le \frac{n}{2}-2$, $\frac{n}{2}+2 \le i \le n-1$.

$$\begin{pmatrix} 2(i-2)-n & \sigma^s(m-1) \\ n-2 & n-2(m+1) \end{pmatrix}$$

We wish to show there are three distinct entries: $n-2 \neq n-2m-2$, $2i-4-n \neq n-2$, $2i-4-n \neq n-2m-2$. Suppose $2i-4-n = n-2 \Rightarrow i=n+1$, which is a contradiction. Suppose $2i-4-n = n-2-2m \Rightarrow i = n-m+2$ and since $m \le \frac{n}{2}-2$ this is a contradiction. Three distinct elements are $\{n-2, n-2m-2, 2i-4-n\}$.

3. G_2 with $m=\frac{n}{2}-1$, $\frac{n}{2}+2 \le i \le n-1$.

$$\begin{pmatrix} 2(i-2)-n & \sigma^s(m-1) \\ n-2 & Y \end{pmatrix}$$

If K *is Even* $\Rightarrow Y=\frac{n}{2}-2$. We wish to show there are three distinct entries: $2i-n-4 \neq n-2$, $n-2 \neq \frac{n}{2}-2$, $2i-n-4 \neq \frac{n}{2}-2$. Suppose $2i-n-4 = n-2 \Rightarrow i=n+1$, which is a contradiction. Suppose $n-2=\frac{n}{2}-2 \Rightarrow n=0$, which is a contradiction. Suppose $2i-n-4=\frac{n}{2}-2 \Rightarrow 4i-4=3n$. We wish to show there are three distinct entries: $2i-n-4 \neq n-2$, $2i-4-n \neq \sigma^s(m-1)$, $n-2 \neq \sigma^s(m-1)$. Suppose $2i-4-n \equiv \sigma^s(m-1) \Rightarrow 6i \equiv 5n+6 \pmod{n-1}$. Subtracting $4i-4=3n$ from this equation $\Rightarrow i \equiv n+1 \pmod{n-1}$, which is a contradiction. Suppose $n-2 \equiv \sigma^s(m-1) \Rightarrow 2i \equiv n+2 \pmod{n-1}$. Subtracting $4i-4=3n \Rightarrow i \equiv n+1 \pmod{n-1}$, which is a contradiction. One of the following two sets has three distinct elements: $\{2i-n-4, n-2, Y\}$ or $\{n-2, \sigma^s(m-1), 2i-4-n\}$.

If K *is Odd* $\Rightarrow Y=\frac{n}{2}+1$. We wish to show there are three distinct entries: $2i-n-4 \neq n-2$, $n-2 \neq \frac{n}{2}+1$, $2i-n-4 \neq \frac{n}{2}+1$. Suppose $2i-n-4 = n-2 \Rightarrow i=n+1$, which is a contradiction. Suppose $n-2=\frac{n}{2}+1 \Rightarrow n=6$, which is a contradiction. Suppose $2i-n-4=\frac{n}{2}+1 \Rightarrow 4i-10=3n$ and since

$n = 6k+6 \Rightarrow 18k = 4i-28 \Rightarrow i = 4.5k+7$ and since k is odd i is not integer, which is a contradiction. There are three distinct elements $\{2i-n-4, n-2, Y\}$.

4. G_2 with $m = \frac{n}{2}-1$, $i = \frac{n}{2}+2$.

$$\begin{pmatrix} Y & \sigma^s(m-1) \\ n-2 & Y \end{pmatrix}$$

If K *is Even* $\Rightarrow Y = \frac{n}{2}-2$. Three distinct elements are $\{Y, n-2, \sigma^s(m-1)\}$. If K *is Odd* $\Rightarrow Y = \frac{n}{2}+1$. We wish to show there are three distinct entries: $Y \neq n-2$, $Y \neq \sigma^s(m-1)$, $n-2 \neq \sigma^s(m-1)$. Suppose $Y \equiv \sigma^s(m-1) \Rightarrow 3n \equiv 8$, which is a contradiction. Suppose $n-2 \equiv \sigma^s(m-1) \Rightarrow n \equiv m \pmod{n-1}$, which is a contradiction. Three distinct elements $\{Y, n-2, \sigma^s(m-1)\}$.

Consider G_3.

1 G_3 with $i = \frac{n}{2}+2$, $j = n$, $l = 1$, $2 \leq m \leq \frac{n-4}{6}$

$$\begin{pmatrix} n-9 & \sigma^s(m-1) \\ n-2 & n-2m \end{pmatrix}$$

There are three distinct entires $\{n-9, n-2, n-2m\}$. Exceptions: For $n = 16$ three distinct entires are $\{5, 14, 16-2m\}$. For $n = 22$ three distinct entires are $\{17, 20, 22-2m\}$.

2 G_3 with $\frac{n}{2}+3 \leq i \leq \frac{5n+4}{6}$, $j = n$, $l = 1$, $2 \leq m \leq \frac{n-4}{6}$.

$$\begin{pmatrix} 2(i-2)-n & \sigma^s(m-1) \\ n-2 & n-2m \end{pmatrix}$$

We wish to show there are three distinct entries: $n-2 \neq n-2m$, $n-2m \neq \sigma^s(m-1)$, $n-2 \neq \sigma^s(m-1)$. Suppose $n-2m \equiv \sigma^s(m-1) \Rightarrow 3m \equiv i \pmod{n-1}$ and this is a contradiction since $i-3m \geq 5$. Suppose $n-2 =\equiv \sigma^s(m-1) \Rightarrow i \equiv m+2$, which is a contradiction. There are three distinct elements $\{n-2, n-2m, \sigma^s(m-1)\}$.

3 G_3 with $\frac{5n+10}{6} \leq i \leq n-1$, $j = n$, $l = 1$, $2 \leq m \leq \frac{n-4}{6}$.

$$\begin{pmatrix} 2(i-1)-n & \sigma^s(m-1) \\ n-2 & n-2m \end{pmatrix}$$

We wish to show there are three distinct entries: $n-2 \neq n-2m$, $n-2m \neq \sigma^s(m-1)$, $n-2 \neq \sigma^s(m-1)$. Suppose $n-2 =\equiv \sigma^s(m-1) \Rightarrow i \equiv m+2 \pmod{n-1}$, which is a contradiction. Suppose $n-2m \equiv \sigma^s(m-1) \Rightarrow 3m \equiv i \pmod{n-1}$ and this is a contradiction since $i-3m \geq \frac{n-1}{3}$. Three distinct elements $\{n-2, n-2m, \sigma^s(m-1)\}$.

4 G_3 with $l = 1$, $\frac{n+2}{6} \leq m \leq \frac{n}{2}-2$, $i = \frac{n}{2}+2$, $j = n$.

$$\begin{pmatrix} n-9 & \sigma^s(m-1) \\ n-2 & n-2(m+1) \end{pmatrix}$$

Three distinct entires are $\{n-9, n-2, n-2(m+1)\}$. Exceptions: For $n = 16$ three distinct entires are $\{5, 14, 14-2m\}$. For $n = 22$ three distinct entires are $\{17, 20, 20-2m\}$.

5 G_3 with $l = 1$, $\frac{n+2}{6} \leq m \leq \frac{n}{2}-2$, $\frac{n}{2}+3 \leq i \leq \frac{5n+4}{6}$, $j = n$

$$\begin{pmatrix} 2(i-2)-n & \sigma^s(m-1) \\ n-2 & n-2(m+1) \end{pmatrix}$$

We wish to show there are three distinct entries: $n-2m-2 \neq n-2$, $n-2 \neq 2i-4-n$, $n-2m-2 \neq 2i-4-n$. Suppose $n-2 = 2i-4-n \Rightarrow i = n+1$, which is a contradiction. Suppose $n-2m-2 = 2i-4-n \Rightarrow m = n-i+1$ and we wish to show there are three distinct entries: $2i-4-n \neq \sigma^s(m-1)$, $n-2 \neq 2i-4-n$, $n-2 \neq \sigma^s(m-1)$. Suppose $2i-4-n = n-2 \Rightarrow i = n+1$, which is a contradiction. Suppose $n-2 \equiv \sigma^s(m-1) \Rightarrow i \equiv m+2 \pmod{n-1}$ and since $i-m \geq 5$ this is a contradiction. Suppose $2i-4-n \equiv \sigma^s(m-1) \Rightarrow m \equiv 3i-2n-4 \pmod{n-1}$. Subtracting this equation from $m = n-i+1 \Rightarrow 3i-2n-4 \equiv n-i+1 \Rightarrow 4i \equiv 3n-5$ and since $n = 6k+4 \Rightarrow i \equiv 4.5k+1.75 \Rightarrow i$ is not integer, which is a contradiction. One of the following two sets has three distinct elements: $\{n-2, n-2m-2, 2i-4-n\}$ or $\{n-2, \sigma^s(m-1), 2i-4-n\}$.

6 G_3 with $l=1$, $\frac{n+2}{6} \leq m \leq \frac{n}{2}-2$, $\frac{5n+10}{6} \leq i \leq n-1$, $j=n$.

$$\begin{pmatrix} 2(i-1)-n & \sigma^s(m-1) \\ n-2 & n-2m-2 \end{pmatrix}$$

We wish to show there are three distinct entries: $n-2 \neq n-2m-2$, $n-2m-2 \neq \sigma^s(m-1)$, $n-2 \neq \sigma^s(m-1)$. Suppose $n-2 =\equiv \sigma^s(m-1) \Rightarrow i \equiv m+2 \pmod{n-1}$ and since $i-m \geq 4$ this is a contradiction. Suppose $n-2m-2 \equiv \sigma^s(m-1) \Rightarrow 3m \equiv i-2 \pmod{n-1}$. We wish to show there are three distinct entries: $n-2 \neq n-2m-2$, $2i-4-n \neq n-2$, $2i-4-n \neq n-2m-2$. Suppose $2i-4-n = n-2m-2 \Rightarrow m = n+1-i$. Subtracting this equation from $m \equiv \frac{i-2}{3} \Rightarrow i \equiv \frac{3n+5}{4} \pmod{n-1}$and since $i \geq \frac{5n+10}{6}$ this is a contradiction. One of the following two sets has three distinct elements: $\{n-2, n-2m-2, \sigma^s(m-1)\}$ or $\{n-2, n-2m-2, 2i-4-n\}$.

7 G_3 with $l=1$, $m = \frac{n}{2}-1$, $i = \frac{n}{2}+2$, $j=n$

$$\begin{pmatrix} n-9 & \sigma^s(m-1) \\ n-2 & n-9 \end{pmatrix}$$

Since $\sigma^s(m-1) \equiv n-3 \pmod{n-1}$ three distinct elements are $(n-2, n-9, \sigma^s(m-1))$ Exceptions: For $n=16$ three distinct entires are $\{14, 5, 13\}$. For $n=22$ three distinct entires are $\{20, 17, 19\}$.

8 G_3 with $m = \frac{n}{2}-1$, $\frac{n}{2}+3 \leq i \leq \frac{5n+4}{6}$, $l=1$

$$\begin{pmatrix} 2(i-2)-n & \sigma^s(m-1) \\ n-2 & n-9 \end{pmatrix}$$

Three distinct elements are $\{2i-n-4, n-2, n-9\}$. Exceptions: For $n=16$ three distinct entires are $\{2i-20, 14, 5\}$. For $n=22$ three distinct entires are $\{2i-26, 20, 17\}$.

9 G_3 with $m = \frac{n}{2}-1$, $\frac{5n+10}{6} \leq i \leq n-1$.

$$\begin{pmatrix} 2(i-\frac{n}{2}-1) & \sigma^s(m-1) \\ n-2 & n-9 \end{pmatrix}$$

Three distinct elements are $\{2i-n-2, n-2, n-9\}$. Exceptions: For $n=16$ three distinct entires are $\{2i-18, 14, 5\}$. For $n=22$ three distinct entires are $\{2i-24, 20, 17\}$.

Case 2: $G(i,j;l,m)$ with $\frac{n}{2}+2 \leq i \leq n-1$, $\frac{n}{2} \leq m \leq n-2$, and let $s=n+1-i$.

$$\begin{pmatrix} X & \sigma^s(m-1) \\ n-2 & n \end{pmatrix}$$

Consider G_1 with $\frac{n}{2}+2 \leq i \leq n-1$, $\frac{n}{2} \leq m \leq n-2$, $X = 2(i-1)-n$. Three distinct entries are $\{n, n-2, 2i-n-2\}$.

Consider G_2.

1. G_2 with $i = \frac{n}{2}+2$. If K *is Even* $\Rightarrow X = \frac{n}{2}-2$, there are three distinct entries $\{n, n-2, X\}$. If K *is Odd* $\Rightarrow X = \frac{n}{2}-1$, there are three distinct entries $\{n, n-2, X\}$.

2. G_2 with $i \geq \frac{n}{2}+3 \Rightarrow X = 2(i-2)-n$. Three distinct entries are $\{n, n-2, X\}$

Consider G_3.

1. G_3 with $i = \frac{n}{2}+2$. There are three distinct entries $\{n, n-2, n-9\}$

2. G_3 with $\frac{n}{2}+3 \leq i \leq \frac{5n+4}{6}$. Three distinct entries $\{n, n-2, 2i-n-4\}$

3. G_3 with $\frac{5n+10}{6} \leq i \leq n-1$. This case is similar to G_2. Three distinct entries are $\{n, n-2, 2i-n-4\}$.

Case 3: $G(i,j;l,m)$ with $\frac{n}{2}+2 \leq i \leq n-1$, $j=n$, $l=1$, $m=n-1$.

$$\begin{pmatrix} X & \sigma^s(m-1) \\ n-2 & n-1 \end{pmatrix}$$

Suppose $\sigma^{n+1-i}(n-2) \equiv n-2 \Rightarrow i+1 \equiv 2n$, which is a contradiction. Suppose $\sigma^{n+1-i}(n-2) \equiv n-1 \Rightarrow n \equiv i$, which is a contradiction. Three distinct elements are $\{n-1, n-2, \sigma^s(m-1)\}$.

Case 4: $G(i,j;l,m)$ with $\frac{n}{2}+2 \leq i \leq n-1$, $j=n$, $l=1$, $m=n$.

$$\begin{pmatrix} X & \sigma^s(m-1) \\ n-2 & 1 \end{pmatrix}$$

Suppose $\sigma^{n+1-i}(n-1) \equiv n-2 \Rightarrow i \equiv n+2$, which is a contradiction. Suppose $\sigma^{n+1-i}(n-1) \equiv 1 \Rightarrow i \equiv 2n-1$, which is a contradiction. Three distinct elements are $\{1, n-2, \sigma^s(m-1)\}$.

$\square$